Building a Typed Scripting Language

Zachary Palmer

The Johns Hopkins University

April 16th, 2015

Scripting: Good and Bad

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print "What number?"
number = raw_input();
if number < 4:
    print "Small!"
else:
    print "Not small!"
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- High-level

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- Flexible
- High-level
- Error-prone

Scripting: Good and Bad

```java
import java.util.*;
public class SmallnessDetector {
    public static void main(String[] arg) {
        Scanner scanner = new Scanner(System.in);
        System.out.println("What number?");
        int number = scanner.nextLine();
        if (number < 4) {
            System.out.println("Small!");
        } else {
            System.out.println("Not small!");
        }
    }
}
```

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```

- Expression has type `String`
- Variable declared with type `int`

The Best of Both Worlds

Why can't we just create a type system?

The Best of Both Worlds

Why can't we just create a type system? We have.

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Rubydust

Flow

MyPy

TeJaS

. . .

Diamondback Ruby

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PHP+QB

Hack

TypeScript

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```python
fs = [str, str.strip, len]
xs = [True, " very ", "ab"]
for (f,x) in zip(fs,xs):
    print f(x)
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```
exec(open(raw_input()).read())
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- Alternative: **build a typed scripting language from scratch**
  - Include "scripty" expressiveness
  - Avoid dynamic operations

This Talk

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Building a Typed Scripting Language

- Primary contribution: synthesis
  - Underused type theory
  - Some abstract interpretation
  - New type theory
- Thesis: *It is possible to construct a language which has the static analyzability of traditional languages and the flexibility of scripting languages.*

Outline

- Duck Type Inference
- Conditional Reasoning
- Contextual Reasoning
- Flexible Data Model
- Formal Development
- What's Left?
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Duck Typing

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Duck typing: categorizing data based on its exhibited properties (as opposed to by explicit grouping).

Java: No Duck Typing

```java
public interface Animal {
    public void speak();
}
public class Dog implements Animal {
    public void speak() {
        System.out.println("Woof!");
    }
}
...
Animal animal = new Dog();
animal.speak();
animal = new Sheep();
animal.speak();
```

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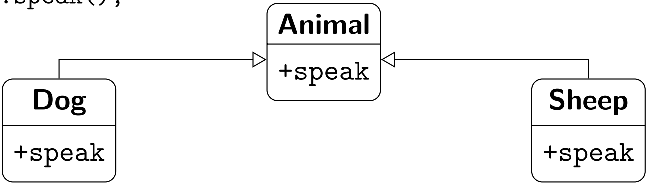

Python: Duck Typing

```python
class Dog:
    def speak(self): print "Woof!"
...
animal = Dog();
animal.speak();
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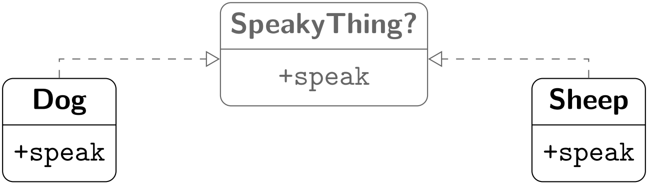

How do we type it?

Constraint Types

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- Expressive:
  - Union types: $\alpha \backslash \{\alpha \geq \texttt{int}\,, \alpha \geq \texttt{char}\}$
  - Recursive types: $\alpha_1 \backslash \{\texttt{nil} \leq \alpha_1, \texttt{cons}\ \ \alpha_2\ \alpha_1 \leq \alpha_1\}$
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  - $\texttt{char} \geq \texttt{int}$ is false, so this type does not exist!

Intuition

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(λx. x>4) (2+1)
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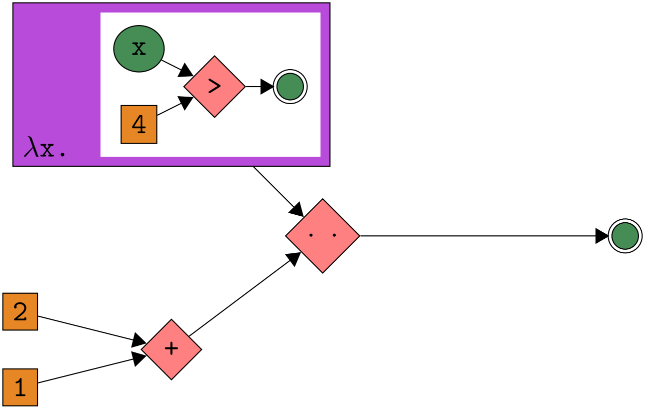

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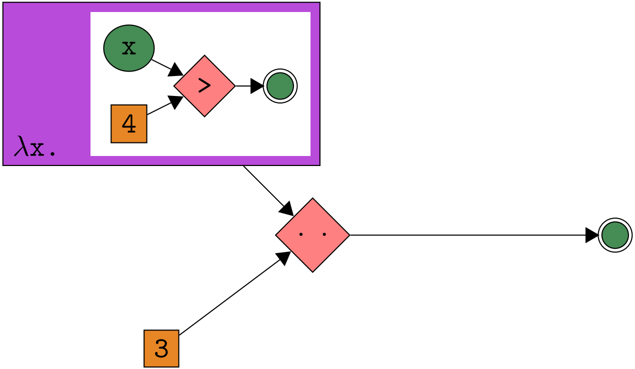

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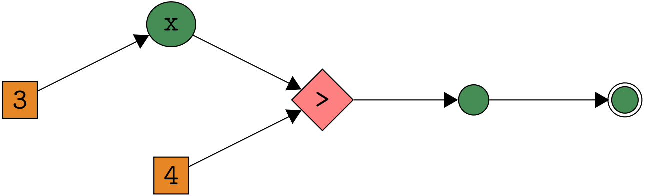

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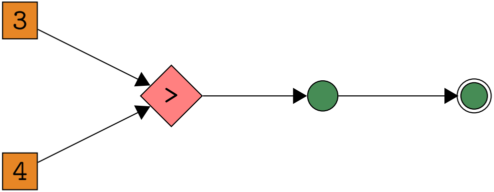

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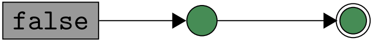

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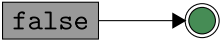

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$$\left\{\begin{array}{l} \alpha_{\mathsf{z}} \geq \alpha_{\mathsf{f}}\ \alpha_{\mathsf{a}}, \\ \alpha_{\mathsf{a}} \geq \texttt{int} \mathbin{\textsf{+}} \texttt{int} \\ \alpha_{\mathsf{f}} \geq \alpha_{\mathsf{x}} \rightarrow \alpha_{\mathsf{r}} \setminus \{\alpha_{\mathsf{r}} \geq \alpha_{\mathsf{x}} \mathbin{\textsf{>}} \texttt{int}\} \end{array}\right\}$$

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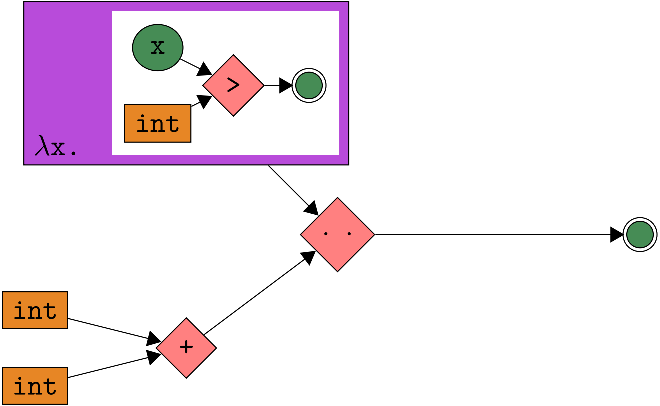

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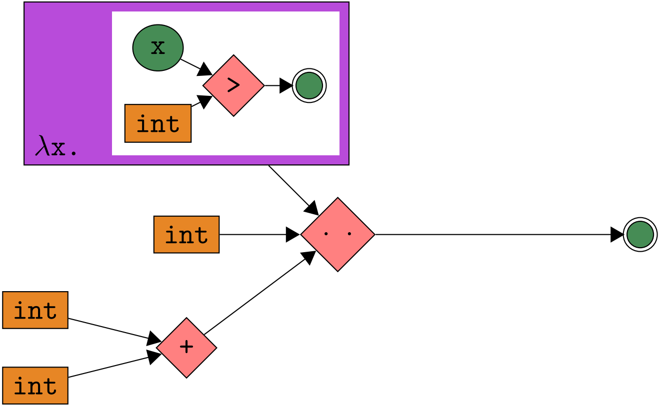

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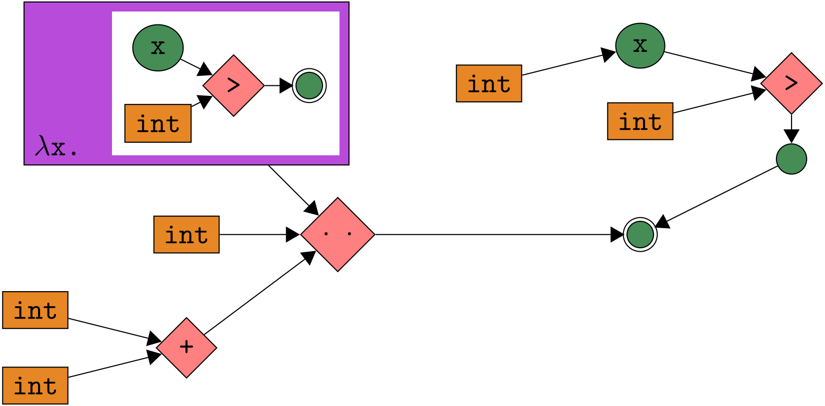

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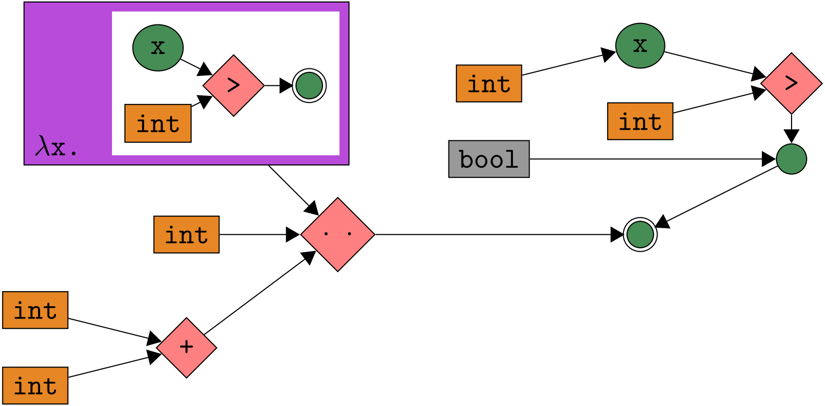

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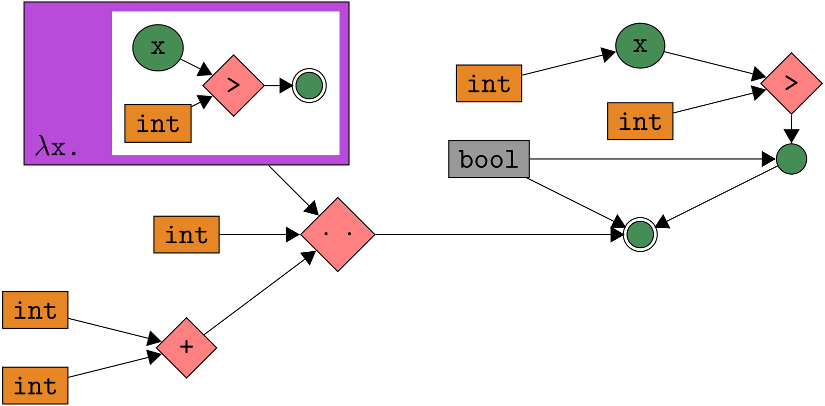

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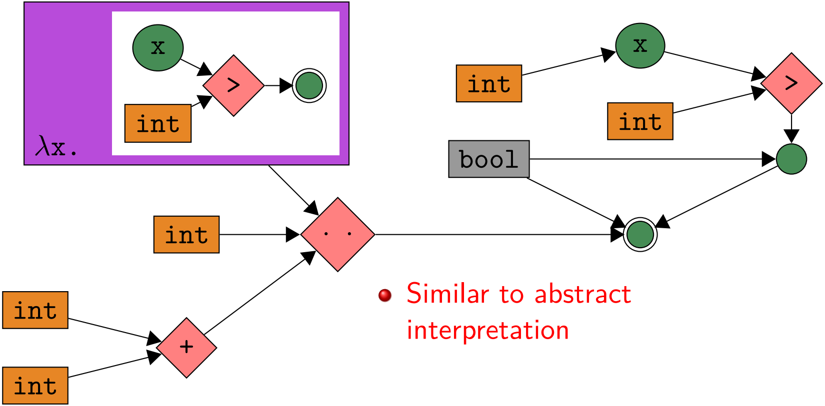

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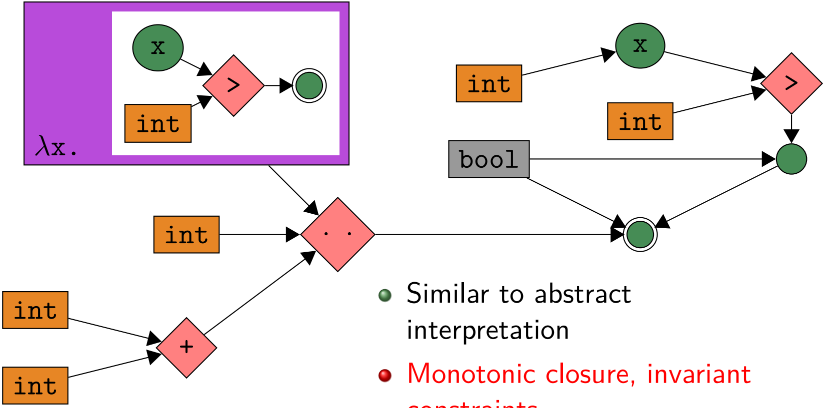

- Similar to abstract interpretation
- Monotonic closure, invariant constraints

Outline

- Duck Type Inference
- Conditional Reasoning
- Contextual Reasoning
- Flexible Data Model
- Formal Development
- What's Left?
- Conclusion

Conditional Reasoning

- Soundness of code based on case analysis

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- Soundness of code based on case analysis
- Common tactic in scripting
- Works particularly well with duck typing

Conditional Reasoning

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def processHooks(tgt, data):
    if callable(tgt):
        fns = [tgt]
    else:
        fns = tgt
    for fn in fns:
        data = fn(data)
    return data
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$\longrightarrow$ function list

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- Program analyses with conditional reasoning: “path-sensitive”

How do we type it?

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- Soundness reasoning by case analysis on value

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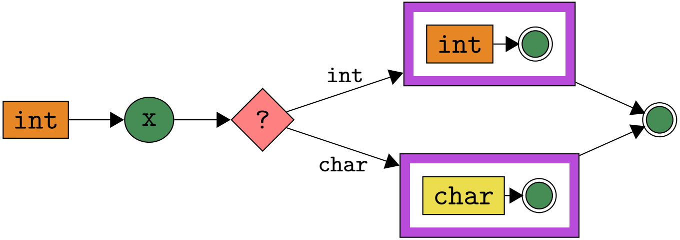

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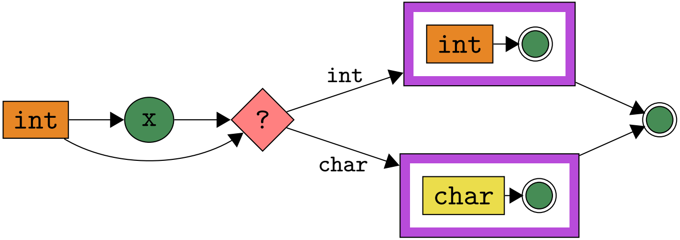

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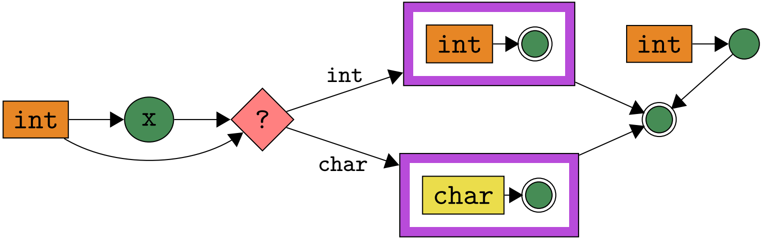

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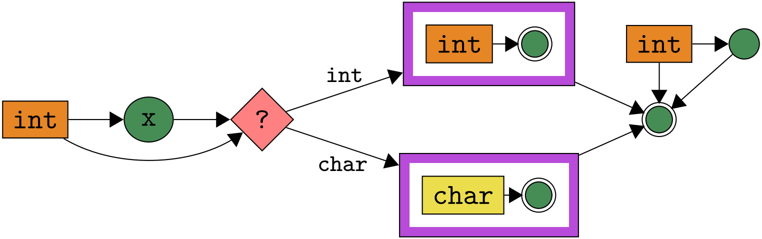

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Filtered Types

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let x = (if somebool then 4 else 'z') in
case x of
    int -> x + 1
    z -> 0
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- We want refinement on case analysis

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- Filtered types: $\tau \cap \pi \cap \pi \cap \ldots$
- Also includes negation (top level only): $\alpha_{\mathtt{z}} \geq 1 - \mathtt{int}$

Filtered Types

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def processHooks(tgt, data):
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Outline

- Duck Type Inference
- Conditional Reasoning
    - Filtered Types
- Contextual Reasoning
- Flexible Data Model
- Formal Development
- What's Left?
- Conclusion

Contextual Reasoning

```
def strange(b):
    if b: return (lambda n: n+1)
    else: return 4
f = strange(True)
x = strange(False)
print f(x)
```

- `f` is a function from integers to integers

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- `x` is a single integer value
- Return types are different based on invocation context
- Program analyses with contextual reasoning: "context-sensitive"

How do we type it?

Context Sensitivity

- Let-bound polymorphism
- Existing program analyses [?]
- Conditional constraints [?]

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- Let-bound polymorphism (too weak)
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Call-Site Polymorphism

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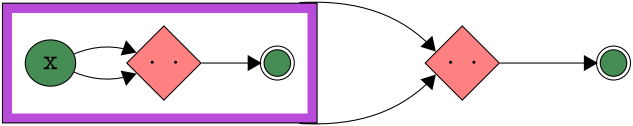
- Polyinstantiation at call sites
- Expressiveness:

```
let f x =
  let g y = y in
  g;;
let h = f();;
let q = (h 1, h 'z');;
```

```
def f() = fun y -> y
def h = f()
def q = (h 1, h 'z')
```

Bounding Call-Site Polymorphism

```
(λx. x x) (λx. x x)
```



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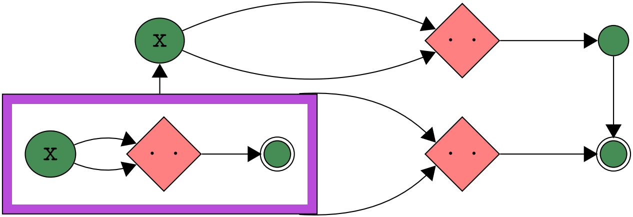

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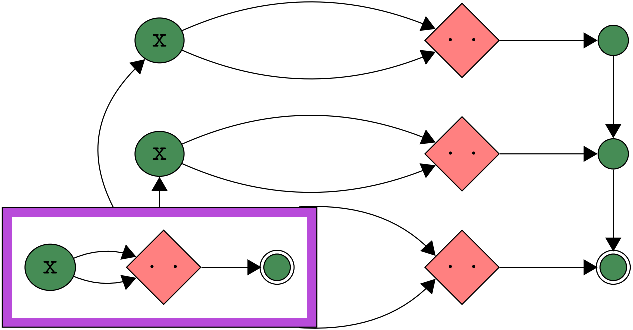

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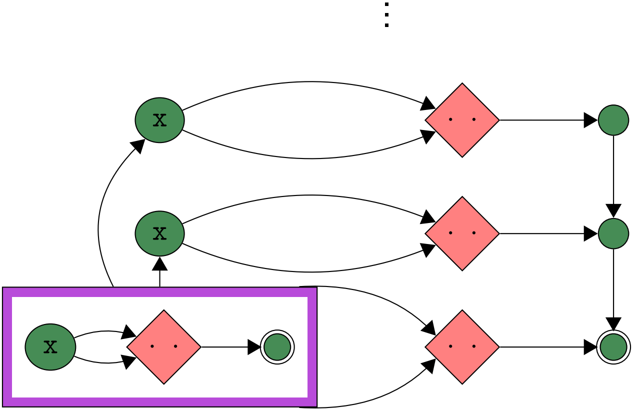

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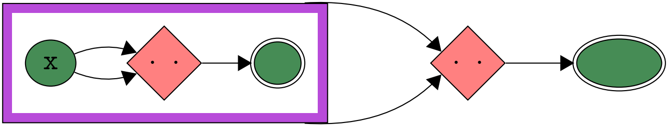

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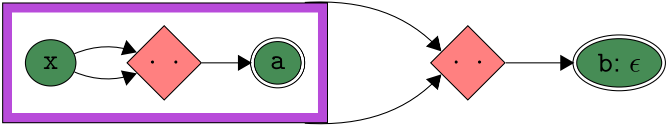

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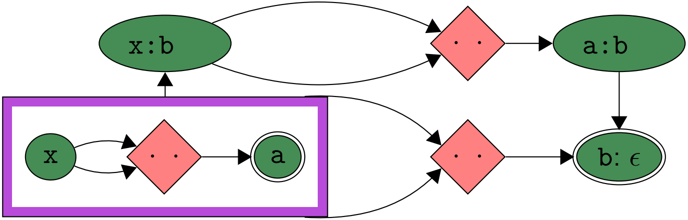
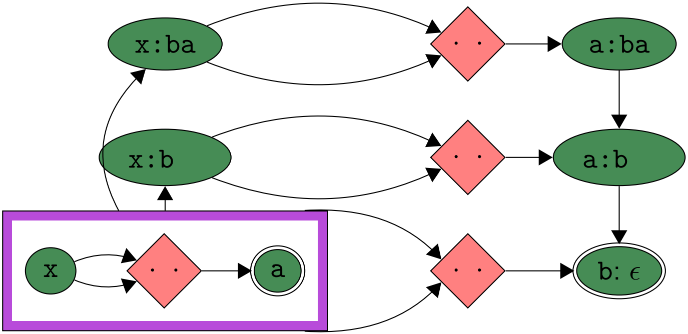


Bounding Call-Site Polymorphism

⋮



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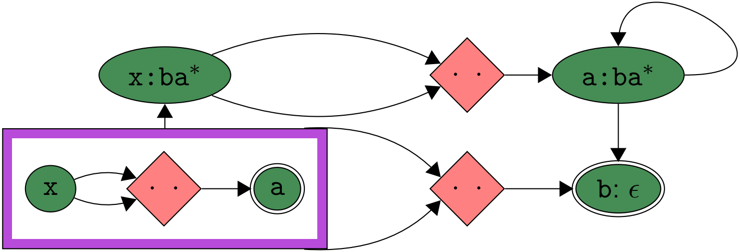
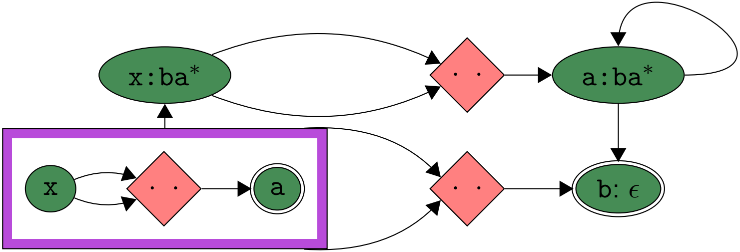


Bounding Call-Site Polymorphism

- Variables named by program point and **contour**



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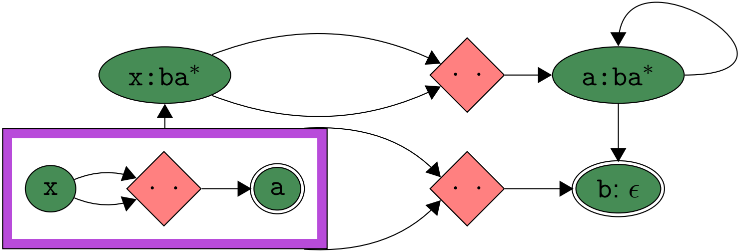

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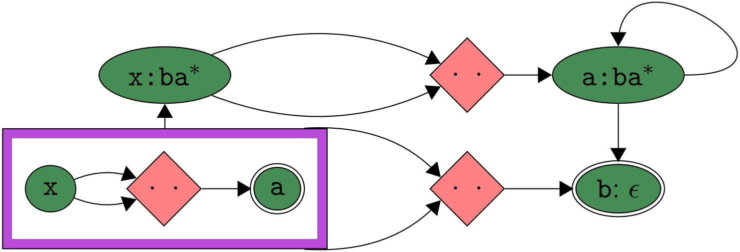

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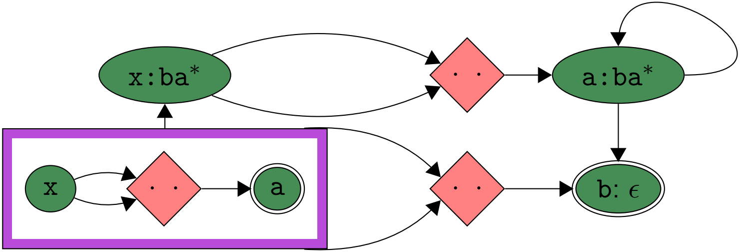

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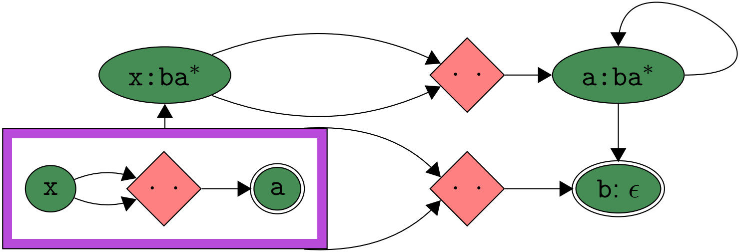

Outline

- Duck Type Inference
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Flexible Data Model

Scripting languages have flexible data models:

```python
class MyClass:
    def msg(self):
        print "Foo"
obj = MyClass()
obj.msg() # Prints "Foo"
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- Mutating/adding methods
- Multiple inheritance
- Dynamic mixins
- etc.

How do we type it?

Encoding

- Reduces complexity of type system

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- Programmer can work "under the hood" as necessary

Encoding

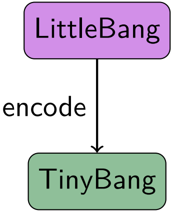
LittleBang

Encoding

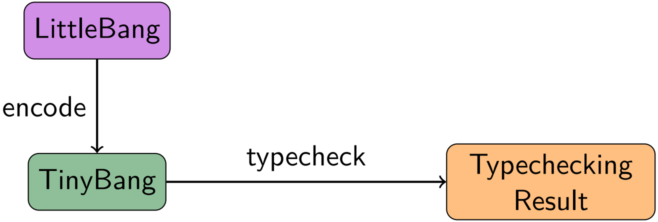
LittleBang

TinyBang

Encoding



Encoding


LittleBang
encode
TinyBang
typecheck
Typechecking Result

TinyBang Language Features

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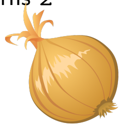
  - Functions match onions by type
    - `(fun ‘Name n -> n) (‘Name "Ann" & ‘Age 43)` returns `"Ann"`
  - **Leftmost** onion element wins
    - `(fun ‘A x -> x) (‘A 2 & ‘B 3 & ‘A 4)` returns 2



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  - `((int -> 0) & (char -> 'a')) ('z')` $\Longrightarrow$ `'a'`
  - `((int -> 0) & (char -> 'a')) (5 & 'z')` $\Longrightarrow$ `0`

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  - `((int -> 0) & (char -> ’a’)) (5 & ’z’)` $\Longrightarrow$ `0`
  - `((int -> 0) & (char -> ’a’)) (‘A 5 & ‘B 4)` crashes

TinyBang Language Features

- Primitives (e.g. `5`)
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- Onions: type-indexed records [**?**]
- Onion dispatch: leftmost function wins
- That's it!

Why These Features?

Together, onions and partial functions can encode:

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- Conditionals (on `'True ()` and `'False ()`)
- Variant-based objects
- Operator overloading
- Classes, inheritance, subclasses, etc.
- Mixins, dynamic functional object extension
- First-class cases
- Optional arguments
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And we can type them!

Encoding Example

LittleBang

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def inc(x,y=1):
    return x + y
print inc(3,5)
print inc(7)
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TinyBang

```
let inc = fun a * `x x ->
            let y = ( (fun `y v -> v)
                    & (fun _ -> 1)     ) a
            in x + y
print (inc (`x 3 & `y 5))
print (inc (`x 7))
```

Working Under the Hood

```
let obj = if somebool
            then object {
                   m(s:str) = print(s)
                 }
            else object {
                   inc(x:int) = x + 1
                 }
obj.m("hello") # static type error if no m

def dynamic(msg): throw MethodError()
(obj & dynamic).m("hello") # exception if no m
```

Working Under the Hood

```
let obj = if somebool
            then
              fun (`msg `m () * `s (s * str)) -> print(s)

            else
              fun (`msg `inc () * `x (x * int)) -> x + 1

obj (`msg `m () & `s "hello") # static type error if no m

let dynamic = fun msg -> throw MethodError()
(obj & dynamic) (`msg `m () & `s "hello") # exception if no m
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TinyBang is:

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- Aware of calling context
- Flexible on data

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def c = 4
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- No mutable monkeypatching (use functional extension!)

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Typechecking Process

To typecheck a LittleBang program:

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- Encode LittleBang operations into TinyBang

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- *A-translate* TinyBang into A-normal form

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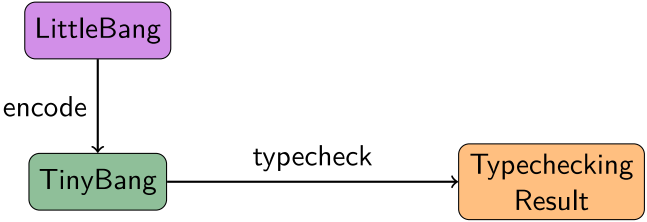

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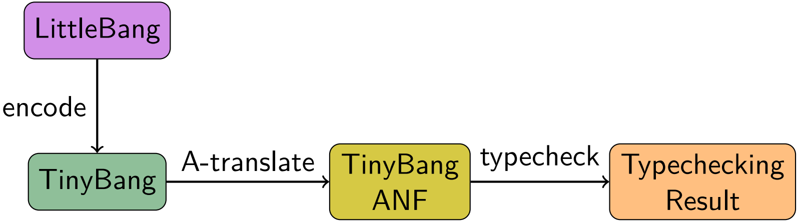

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- Check consistency of result
  - If the program is bad, we will reach a false conclusion

Synthesis

- TinyBang type grammar is shallow (unusual)
  - e.g. $\alpha_1 \backslash \{ \texttt{`A}\, \alpha_2 \leq \alpha_1, \texttt{int} \leq \alpha_2 \}$
  - not `` `A int ``

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  - value $\leftrightarrow$ type
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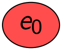
Proof of Soundness

Progress & Preservation



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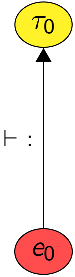

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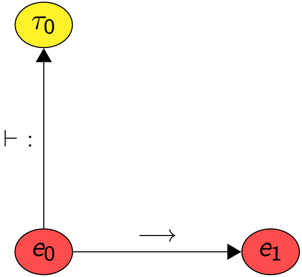

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$$\begin{array}{ccc} \tau_0 & & \tau_1 \\ \uparrow{\scriptstyle \vdash :} & & \uparrow{\scriptstyle \vdash :} \\ e_0 & \xrightarrow{\longrightarrow} & e_1 \end{array}$$

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\begin{tikzcd}
\tau_0 \arrow[r, dashed, "\geq"] & \tau_1 \\
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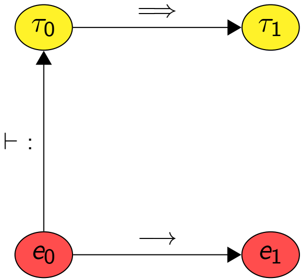

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- Novel object extension properties [**?**]

Outline

- Duck Type Inference
- Conditional Reasoning
  - Filtered Types
- Contextual Reasoning
- Flexible Data Model
- Formal Development
- What's Left?
- Conclusion

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- Current Research

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- Exploit connection to abstract interpretation while remaining in type theory

Thanks!

- Scott F. Smith (advisor)
- Alexander Rozenstheyn (collaborator)
- Pottayil Harisanker Menon (collaborator)
- Rebekah Palmer (wife, best friend)
- JHU Computer Science Department
- All those people who did all that research

Questions?

Bibliography

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Typechecking by Example

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Encoding and A-normalization

Preparing to Typecheck

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TinyBang ANF

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x3 = + x1 x2;
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x6 = x4 & x5;
x7 = x6 b;
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Initial Alignment

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x3 = + x1 x2;
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- Replace primitive data with its type

Initial Alignment

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- Replace primitive data with its type
- Convert to constraints (for e.g. duck typing)

Perform Constraint Closure

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\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\,\&\,\alpha_{\mathtt{x5}}, \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \\
\\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\texttt{b}} \geq \texttt{`True}\ \ldots, \quad \alpha_{\texttt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\texttt{x1}} \geq \texttt{int}, & \\
\alpha_{\texttt{x2}} \geq \texttt{int}, & \\
\alpha_{\texttt{x3}} \geq \texttt{+}\ \alpha_{\texttt{x1}}\ \alpha_{\texttt{x2}}, & \alpha_{\texttt{x3}} \geq \texttt{int}, \\
\alpha_{\texttt{x4}} \geq \{\alpha_{\texttt{p1}} \geq \texttt{()}\,, \alpha_{\texttt{p2}} \geq \texttt{`True}\,\alpha_{\texttt{p1}}\} \rightarrow \{\alpha_{\texttt{r1}} \geq \texttt{int}\}, & \\
\alpha_{\texttt{x5}} \geq \{\alpha_{\texttt{p3}} \geq \texttt{()}\,, \alpha_{\texttt{p4}} \geq \texttt{`False}\,\alpha_{\texttt{p3}}\} \rightarrow \{\alpha_{\texttt{r2}} \geq \texttt{int}\}, & \\
\alpha_{\texttt{x6}} \geq \alpha_{\texttt{x4}} \,\&\, \alpha_{\texttt{x5}}, & \\
\alpha_{\texttt{x7}} \geq \alpha_{\texttt{x6}}\ \alpha_{\texttt{b}}, & \\
& \\
\alpha_{\texttt{x8}} \geq \texttt{+}\ \alpha_{\texttt{x3}}\ \alpha_{\texttt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\ \&\ \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\ \&\ \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$
\left\{
\begin{array}{ll}
\alpha_{\texttt{b}} \geq \texttt{`True}\ \ldots, \qquad \alpha_{\texttt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\texttt{x1}} \geq \texttt{int}, & \\
\alpha_{\texttt{x2}} \geq \texttt{int}, & \\
\alpha_{\texttt{x3}} \geq \texttt{+}\ \alpha_{\texttt{x1}}\ \alpha_{\texttt{x2}}, & \alpha_{\texttt{x3}} \geq \texttt{int}, \\
\alpha_{\texttt{x4}} \geq \{\alpha_{\texttt{p1}} \geq \texttt{()}\,, \alpha_{\texttt{p2}} \geq \texttt{`True}\,\alpha_{\texttt{p1}}\} \rightarrow \{\alpha_{\texttt{r1}} \geq \texttt{int}\}, & \alpha_{\texttt{r1'}} \geq \texttt{int}, \\
\alpha_{\texttt{x5}} \geq \{\alpha_{\texttt{p3}} \geq \texttt{()}\,, \alpha_{\texttt{p4}} \geq \texttt{`False}\,\alpha_{\texttt{p3}}\} \rightarrow \{\alpha_{\texttt{r2}} \geq \texttt{int}\}, & \\
\alpha_{\texttt{x6}} \geq \alpha_{\texttt{x4}}\,\&\,\alpha_{\texttt{x5}}, & \\
\alpha_{\texttt{x7}} \geq \alpha_{\texttt{x6}}\ \alpha_{\texttt{b}}, & \alpha_{\texttt{x7}} \geq \alpha_{\texttt{r1'}}, \\
 & \\
\alpha_{\texttt{x8}} \geq \texttt{+}\ \alpha_{\texttt{x3}}\ \alpha_{\texttt{x7}} &
\end{array}
\right\}
$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \texttt{`True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \texttt{int}, & \\
\alpha_{\mathtt{x2}} \geq \texttt{int}, & \\
\alpha_{\mathtt{x3}} \geq \texttt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \texttt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \texttt{()}\,, \alpha_{\mathtt{p2}} \geq \texttt{`True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \texttt{int}\}, & \alpha_{\mathtt{r1'}} \geq \texttt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \texttt{()}\,, \alpha_{\mathtt{p4}} \geq \texttt{`False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \texttt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \\
\alpha_{\mathtt{x8}} \geq \texttt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\ \&\ \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\ , \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\, \alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\ , \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\, \alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\ \&\ \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
& \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \texttt{`True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \texttt{int}, & \\
\alpha_{\mathtt{x2}} \geq \texttt{int}, & \\
\alpha_{\mathtt{x3}} \geq \texttt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \texttt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \texttt{()}\,, \alpha_{\mathtt{p2}} \geq \texttt{`True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \texttt{int}\}, & \alpha_{\mathtt{r1'}} \geq \texttt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \texttt{()}\,, \alpha_{\mathtt{p4}} \geq \texttt{`False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \texttt{int}\}, & \alpha_{\mathtt{r2'}} \geq \texttt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \texttt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\,\&\,\alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \quad \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \texttt{`True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \texttt{int}, & \\
\alpha_{\mathtt{x2}} \geq \texttt{int}, & \\
\alpha_{\mathtt{x3}} \geq \texttt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \texttt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \texttt{()}\,, \alpha_{\mathtt{p2}} \geq \texttt{`True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \texttt{int}\}, & \alpha_{\mathtt{r1'}} \geq \texttt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \texttt{()}\,, \alpha_{\mathtt{p4}} \geq \texttt{`False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \texttt{int}\}, & \alpha_{\mathtt{r2'}} \geq \texttt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \qquad \alpha_{\mathtt{x7}} \geq \texttt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \texttt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \qquad \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$
\left\{
\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \to \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r1}'} \geq \mathtt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \to \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r2}'} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \mathbin{\&} \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \quad \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1}'}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2}'}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}, & \alpha_{\mathtt{x8}} \geq \mathtt{int}
\end{array}
\right\}
$$

Check Consistency

$$
\left\{
\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \mathtt{`True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \mathtt{`False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \mathtt{`True}\,\alpha_{\mathtt{p1}}\} \to \{\alpha_{\mathtt{r1}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \mathtt{`False}\,\alpha_{\mathtt{p3}}\} \to \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \qquad \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}, & \alpha_{\mathtt{x8}} \geq \mathtt{int}
\end{array}
\right\}
$$

✓

Typechecking by Example

First, we will typecheck this program:

```
let b = ...
1 + 2 + (if b then 5 else 1)
```

✓

Then, we will typecheck this program:

```
let b = ...
1 + 2 + (if b then 'z' else 1)
```

Initial Alignment

```
b = ...;
x1 = 1;
x2 = 2;
x3 = + x1 x2;
x4 = { p1 = (); p2 = ‘True p1 } -> { r1 = ’z’ };
x5 = { p3 = (); p4 = ‘False p3 } -> { r2 = 1 };
x6 = x4 & x5;
x7 = x6 b;
x8 = + x3 x7;
```

Initial Alignment

$$\left\{\begin{array}{l}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}\,, \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}\,, \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\, \alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\, \alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}},
\end{array}\right\}$$

- Same as last time, but with a `char`

Perform Constraint Closure

$$
\left\{
\begin{array}{l}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \to \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \to \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \\
\\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}
\end{array}
\right\}
$$

Perform Constraint Closure

$$\left\{\begin{array}{l}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \\
\\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\texttt{b}} \geq \texttt{`True}\ \ldots, \quad \alpha_{\texttt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\texttt{x1}} \geq \texttt{int}, & \\
\alpha_{\texttt{x2}} \geq \texttt{int}, & \\
\alpha_{\texttt{x3}} \geq \texttt{+}\ \alpha_{\texttt{x1}}\ \alpha_{\texttt{x2}}, & \alpha_{\texttt{x3}} \geq \texttt{int}, \\
\alpha_{\texttt{x4}} \geq \{\alpha_{\texttt{p1}} \geq \texttt{()}, \alpha_{\texttt{p2}} \geq \texttt{`True}\,\alpha_{\texttt{p1}}\} \rightarrow \{\alpha_{\texttt{r1}} \geq \texttt{char}\}, & \\
\alpha_{\texttt{x5}} \geq \{\alpha_{\texttt{p3}} \geq \texttt{()}, \alpha_{\texttt{p4}} \geq \texttt{`False}\,\alpha_{\texttt{p3}}\} \rightarrow \{\alpha_{\texttt{r2}} \geq \texttt{int}\}, & \\
\alpha_{\texttt{x6}} \geq \alpha_{\texttt{x4}} \,\&\, \alpha_{\texttt{x5}}, & \\
\alpha_{\texttt{x7}} \geq \alpha_{\texttt{x6}}\ \alpha_{\texttt{b}}, & \\
 & \\
\alpha_{\texttt{x8}} \geq \texttt{+}\ \alpha_{\texttt{x3}}\ \alpha_{\texttt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$
\left\{
\begin{array}{lr}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq +\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq ()\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\, \alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq ()\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\, \alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq +\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}
\right\}
$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, \qquad \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq +\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq ()\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq ()\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq +\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\, \alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\, \alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\, \&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\ \&\ \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}\right\}$$

Perform Constraint Closure

$$
\left\{
\begin{array}{ll}
\alpha_{\texttt{b}} \geq \text{`}\texttt{True}\ \ldots, \qquad \alpha_{\texttt{b}} \geq \text{`}\texttt{False}\ \ldots, & \\
\alpha_{\texttt{x1}} \geq \texttt{int}, & \\
\alpha_{\texttt{x2}} \geq \texttt{int}, & \\
\alpha_{\texttt{x3}} \geq \texttt{+}\ \alpha_{\texttt{x1}}\ \alpha_{\texttt{x2}}, & \alpha_{\texttt{x3}} \geq \texttt{int}, \\
\alpha_{\texttt{x4}} \geq \{\alpha_{\texttt{p1}} \geq \texttt{()}\,, \alpha_{\texttt{p2}} \geq \text{`}\texttt{True}\,\alpha_{\texttt{p1}}\} \rightarrow \{\alpha_{\texttt{r1}} \geq \texttt{char}\}, & \alpha_{\texttt{r1'}} \geq \texttt{char}, \\
\alpha_{\texttt{x5}} \geq \{\alpha_{\texttt{p3}} \geq \texttt{()}\,, \alpha_{\texttt{p4}} \geq \text{`}\texttt{False}\,\alpha_{\texttt{p3}}\} \rightarrow \{\alpha_{\texttt{r2}} \geq \texttt{int}\}, & \\
\alpha_{\texttt{x6}} \geq \alpha_{\texttt{x4}}\,\&\,\alpha_{\texttt{x5}}, & \\
\alpha_{\texttt{x7}} \geq \alpha_{\texttt{x6}}\ \alpha_{\texttt{b}}, & \alpha_{\texttt{x7}} \geq \alpha_{\texttt{r1'}}, \\
 & \\
\alpha_{\texttt{x8}} \geq \texttt{+}\ \alpha_{\texttt{x3}}\ \alpha_{\texttt{x7}} &
\end{array}
\right\}
$$

Perform Constraint Closure

$$
\left\{
\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\ , \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\, \alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\ , \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\, \alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} &
\end{array}
\right\}
$$

Perform Constraint Closure

$$
\left\{
\begin{array}{ll}
\alpha_{\texttt{b}} \geq \texttt{`True}\ \ldots, \qquad \alpha_{\texttt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\texttt{x1}} \geq \texttt{int}, & \\
\alpha_{\texttt{x2}} \geq \texttt{int}, & \\
\alpha_{\texttt{x3}} \geq \texttt{+}\ \alpha_{\texttt{x1}}\ \alpha_{\texttt{x2}}, & \alpha_{\texttt{x3}} \geq \texttt{int}, \\
\alpha_{\texttt{x4}} \geq \{\alpha_{\texttt{p1}} \geq \texttt{()}\,, \alpha_{\texttt{p2}} \geq \texttt{`True}\,\alpha_{\texttt{p1}}\} \rightarrow \{\alpha_{\texttt{r1}} \geq \texttt{char}\}, & \alpha_{\texttt{r1'}} \geq \texttt{char}, \\
\alpha_{\texttt{x5}} \geq \{\alpha_{\texttt{p3}} \geq \texttt{()}\,, \alpha_{\texttt{p4}} \geq \texttt{`False}\,\alpha_{\texttt{p3}}\} \rightarrow \{\alpha_{\texttt{r2}} \geq \texttt{int}\}, & \alpha_{\texttt{r2'}} \geq \texttt{int}, \\
\alpha_{\texttt{x6}} \geq \alpha_{\texttt{x4}}\,\&\,\alpha_{\texttt{x5}}, & \\
\alpha_{\texttt{x7}} \geq \alpha_{\texttt{x6}}\ \alpha_{\texttt{b}}, & \alpha_{\texttt{x7}} \geq \alpha_{\texttt{r1'}}, \\
 & \alpha_{\texttt{x7}} \geq \alpha_{\texttt{r2'}}, \\
\alpha_{\texttt{x8}} \geq \texttt{+}\ \alpha_{\texttt{x3}}\ \alpha_{\texttt{x7}} &
\end{array}
\right\}
$$

Perform Constraint Closure

$$
\left\{
\begin{array}{lll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\,\&\,\alpha_{\mathtt{x5}}, & & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \quad \alpha_{\mathtt{x7}} \geq \mathtt{char}, & & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} & &
\end{array}
\right\}
$$

Perform Constraint Closure

$$\left\{\begin{array}{llll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, & \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & & & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & & & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & & & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & & & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & & & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & & & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \mathtt{char}, & \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & & & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} & & &
\end{array}\right\}$$

Perform Constraint Closure

$$\left\{\begin{array}{llll}
\alpha_{\mathtt{b}} \geq \text{‘}\mathtt{True}\ \ldots, & \alpha_{\mathtt{b}} \geq \text{‘}\mathtt{False}\ \ldots, & & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & & & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & & & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & & & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{‘}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & & & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{‘}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & & & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & & & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \mathtt{char}, & \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & & & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}} & & &
\end{array}\right\}$$

Perform Constraint Closure

$$
\left\{
\begin{array}{ll}
\alpha_{\texttt{b}} \geq \texttt{`True}\ \ldots, \quad \alpha_{\texttt{b}} \geq \texttt{`False}\ \ldots, & \\
\alpha_{\texttt{x1}} \geq \texttt{int}, & \\
\alpha_{\texttt{x2}} \geq \texttt{int}, & \\
\alpha_{\texttt{x3}} \geq \texttt{+}\ \alpha_{\texttt{x1}}\ \alpha_{\texttt{x2}}, & \alpha_{\texttt{x3}} \geq \texttt{int}, \\
\alpha_{\texttt{x4}} \geq \{\alpha_{\texttt{p1}} \geq \texttt{()}\,, \alpha_{\texttt{p2}} \geq \texttt{`True}\, \alpha_{\texttt{p1}}\} \rightarrow \{\alpha_{\texttt{r1}} \geq \texttt{char}\}, & \alpha_{\texttt{r1'}} \geq \texttt{char}, \\
\alpha_{\texttt{x5}} \geq \{\alpha_{\texttt{p3}} \geq \texttt{()}\,, \alpha_{\texttt{p4}} \geq \texttt{`False}\, \alpha_{\texttt{p3}}\} \rightarrow \{\alpha_{\texttt{r2}} \geq \texttt{int}\}, & \alpha_{\texttt{r2'}} \geq \texttt{int}, \\
\alpha_{\texttt{x6}} \geq \alpha_{\texttt{x4}} \,\&\, \alpha_{\texttt{x5}}, & \\
\alpha_{\texttt{x7}} \geq \alpha_{\texttt{x6}}\ \alpha_{\texttt{b}}, \quad \alpha_{\texttt{x7}} \geq \texttt{char}, \quad \alpha_{\texttt{x7}} \geq \texttt{int}, & \alpha_{\texttt{x7}} \geq \alpha_{\texttt{r1'}}, \\
 & \alpha_{\texttt{x7}} \geq \alpha_{\texttt{r2'}}, \\
\alpha_{\texttt{x8}} \geq \texttt{+}\ \alpha_{\texttt{x3}}\ \alpha_{\texttt{x7}}, & \alpha_{\texttt{x8}} \geq \texttt{int}
\end{array}
\right\}
$$

Check Consistency

$$
\left\{
\begin{array}{llll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, & \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & & & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & & & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & & & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\multicolumn{3}{l}{\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\ , \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\, \alpha_{\mathtt{p1}}\} \rightarrow \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\},} & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\multicolumn{3}{l}{\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\ , \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\, \alpha_{\mathtt{p3}}\} \rightarrow \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\},} & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}}\, \&\, \alpha_{\mathtt{x5}}, & & & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, & \alpha_{\mathtt{x7}} \geq \mathtt{char}, & \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & & & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}, & & & \alpha_{\mathtt{x8}} \geq \mathtt{int}
\end{array}
\right\}
$$

Check Consistency

$$
\left\{
\begin{array}{ll}
\alpha_{\mathtt{b}} \geq \text{`}\mathtt{True}\ \ldots, \quad \alpha_{\mathtt{b}} \geq \text{`}\mathtt{False}\ \ldots, & \\
\alpha_{\mathtt{x1}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x2}} \geq \mathtt{int}, & \\
\alpha_{\mathtt{x3}} \geq \mathtt{+}\ \alpha_{\mathtt{x1}}\ \alpha_{\mathtt{x2}}, & \alpha_{\mathtt{x3}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x4}} \geq \{\alpha_{\mathtt{p1}} \geq \mathtt{()}\,, \alpha_{\mathtt{p2}} \geq \text{`}\mathtt{True}\,\alpha_{\mathtt{p1}}\} \to \{\alpha_{\mathtt{r1}} \geq \mathtt{char}\}, & \alpha_{\mathtt{r1'}} \geq \mathtt{char}, \\
\alpha_{\mathtt{x5}} \geq \{\alpha_{\mathtt{p3}} \geq \mathtt{()}\,, \alpha_{\mathtt{p4}} \geq \text{`}\mathtt{False}\,\alpha_{\mathtt{p3}}\} \to \{\alpha_{\mathtt{r2}} \geq \mathtt{int}\}, & \alpha_{\mathtt{r2'}} \geq \mathtt{int}, \\
\alpha_{\mathtt{x6}} \geq \alpha_{\mathtt{x4}} \,\&\, \alpha_{\mathtt{x5}}, & \\
\alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{x6}}\ \alpha_{\mathtt{b}}, \quad \alpha_{\mathtt{x7}} \geq \mathtt{char}, \quad \alpha_{\mathtt{x7}} \geq \mathtt{int}, & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r1'}}, \\
 & \alpha_{\mathtt{x7}} \geq \alpha_{\mathtt{r2'}}, \\
\alpha_{\mathtt{x8}} \geq \mathtt{+}\ \alpha_{\mathtt{x3}}\ \alpha_{\mathtt{x7}}, & \alpha_{\mathtt{x8}} \geq \mathtt{int}
\end{array}
\right\}
$$

✗

Typechecking by Example

First, we will typecheck this program:

```
let b = ...
1 + 2 + (if b then 5 else 1)
```

✓

Then, we will typecheck this program:

```
let b = ...
1 + 2 + (if b then 'z' else 1)
```

✗