$\boxed{\text{Programs}}$ (2 weeks)

$\boxed{\text{Proofs about Programs}}$ (until now) $e \Rightarrow v$ $\Gamma \vdash e : \tau$

$\boxed{\text{Proofs about Proofs about Programs}}$ (from today)

Operational Equivalence

"these two expressions are interchangable"

$\not\cong$ (Function $x \rightarrow x$) $\qquad$ $'a \rightarrow 'a$
(Function $x \rightarrow x - 1 + 1$) $\qquad$ Int$\rightarrow$Int

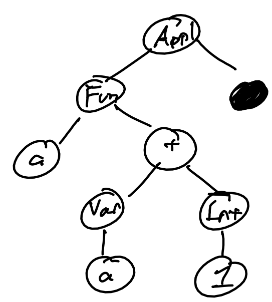
(Function $x \rightarrow x+1$) $\cong$ (Function $x \rightarrow x+1-1+1$)

"context" C — an AST with one hole ($\bullet$) in it

$\boxed{\underset{\text{operational equivalence}}{e_1 \cong e_2}}$ iff $\boxed{\forall C.\ C[e_1] \Rightarrow v_1 \text{ if and only if } C[e_2] \Rightarrow v_2}$

Contexts

(Function $a \rightarrow a+1$) $\bullet$



$\bullet + 2$

(Function $a \rightarrow \bullet\ a$) 5

0 0 $\bullet$

General proof advice: to prove $\forall$, pretend you've picked one but you don't know which one you've picked

to disprove $\forall$, just find one counterexample

Examples of inequivalence

$\forall C.\ C[e_1] \Rightarrow v_1$ iff $C[e_2] \Rightarrow v_2$

$4 \ncong 5$

$C[4] \Rightarrow 4$

$C[5] \not\Rightarrow$

$C = \boxed{\begin{array}{l}\text{Let } f = \text{Function self} \to \text{Function } x \to \\ \quad \text{If } x=4 \text{ Then } x \text{ Else} \\ \quad\quad \text{self self } x \\ \text{In} \\ f\ f\ \bullet\end{array}}$

$\boxed{\text{If } \bullet = 4 \text{ Then } 0\ \ 0 \text{ Else } 0}$

$e ::= $ (same as Fb)

$v ::= $ (same as Fb)

opsem for "F-"

$e \Rightarrow 0$

$a \ncong \text{Not (Not } a)$

$\boxed{(\text{Function } a \to \bullet + 1)\ \ 1}$

For operational equiv., operational semantics matter.

$C[a] \Rightarrow 2$

$C[\text{Not(Not } a)] \not\Rightarrow$

$(\text{Function } a \to b) \ncong (\text{Function } a \to c)$

$\boxed{(\text{Function } b \to \bullet)\ \ 5}$

$C[\text{Function } a \to b] = (\text{Function } b \to \text{Function } a \to b)\ 5 \Rightarrow \text{Function } a \to 5$

$C[\text{Function } a \to c] = (\text{Function } b \to \text{Function } a \to c)\ 5 \not\Rightarrow$

Example equivalences

$\forall C.\ C[e_1] \Rightarrow v_1$ iff $C[e_2] \Rightarrow v_2$

- Reflexivity — $e \cong e$
- Symmetry — If $e \cong e'$ then $e' \cong e$
- Transitivity — If $e_1 \cong e_2$ and $e_2 \cong e_3$ then $e_1 \cong e_3$.
- Congruence — If $e \cong e'$ then $C[e] \cong C[e']$ for any C.
- α-equivalence — Renaming bound variables produces equivalent programs.

  ex. $\text{Function } a \to a \cong \text{Function } b \to b$

  $\text{Function } a \to a + b \ncong \text{Function } b \to b + b$
- β-equivalence — Taking a single step of execution produces equivalent expressions.

  ex. $1 + 2 \cong 3$
- η-equivalence — $e \cong (\text{Function } x \to e)\ v$ for any x not free in e

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$e_1 = \text{Function } a \to a$

$e_2 = \text{Function } b \to b$

$C = \text{Function } a \to \bullet$

$C[e_1] = \text{Function } a \to \text{Function } a \to a$

$C[e_2] = \text{Function } a \to \text{Function } b \to b$

$e_1 \cong e_2$ defined as $\forall C.\ C[e_1] \Rightarrow v_1$ iff $C[e_2] \Rightarrow v_2$

If $e_1 \cong e_2$ then $\forall C.\ C[e_1] \Rightarrow v$ iff $C[e_2] \Rightarrow v$

Proof by contradiction. Suppose for some $e_1 \cong e_2$ that, in some ctx C, $C[e_1] \Rightarrow v_1$ and $C[e_2] \Rightarrow v_2$ s.t. $v_1 \neq v_2$. Then we can create ctx C' s.t. $C'[e_1] \Rightarrow v_1'$ and $C'[e_2] \not\Rightarrow$.

- If v_1 and v_2 are both integers, then C' is "If $C = v_1$ Then 0 Else 0 0"
- ⋮

Metaproof: Proof about proofs