TFb Soundness

If $e \Rightarrow v$ and $\Gamma \vdash e : \tau$ then $\Gamma \vdash v : \tau$..

True $\dfrac{}{\Gamma \vdash \text{True} : \text{Bool}}$ $\qquad$ + $\dfrac{\Gamma \vdash e_1 : \text{Int} \quad \Gamma \vdash e_2 : \text{Int}}{\Gamma \vdash e_1 + e_2 : \text{Int}}$

$\cancel{\overset{!!}{\frown} \dfrac{}{\Gamma \vdash 4 : \text{Bool}}}$ $\qquad$ $\cancel{\underset{\frown}{\text{And}^{!!}} \dfrac{\Gamma \vdash e_1 : \tau_1 \to \tau_2}{\Gamma \vdash e_1 \text{ And } e_2 : \text{Int}}}$

$\Gamma \vdash e : \tau$ and x not free in e then $\Gamma, x:\tau' \vdash e : \tau$

By induction on height of $\Gamma \vdash e : \tau$ then by case analysis on the rule used.

Case of Int Rule: $\dfrac{}{\Gamma \vdash n : \text{Int}}$ so by Int Rule $\dfrac{}{\Gamma, x:\tau' \vdash n : \text{Int}}$

Case of + Rule: $\dfrac{\Gamma \vdash e_1 : \text{Int} \quad \Gamma \vdash e_2 : \text{Int}}{\Gamma \vdash e_1 + e_2 : \text{Int}}$ By Ind, $\Gamma, x:\tau' \vdash e_1 : \text{Int}$

$\Gamma, x:\tau' \vdash e_2 : \text{Int}$

By + Rule:

$\Gamma, x:\tau' \vdash e_1 + e_2 : \text{Int}$

Case of Let: $\dfrac{\Gamma \vdash e_1 : \tau'' \quad \Gamma, x':\tau'' \vdash e_2 : \tau}{\Gamma \vdash (\text{Let } x':\tau'' = e_1 \text{ In } e_2) : \tau}$

By ind. $\Gamma, x:\tau' \vdash e_1 : \tau''$.

Two subcases: $x = x'$ or $x \neq x'$

Let

$$\dfrac{\Gamma, x:\tau' \vdash e_1 : \tau'' \quad \Gamma, x:\tau', x':\tau'' \vdash e_2 : \tau}{\Gamma, x:\tau' \vdash (\text{Let } x':\tau'' = e_1 \text{ In } e_2) : \tau}$$

Case $x \neq x'$: By ind: $\boxed{\Gamma, x':\tau'', x:\tau' \vdash e_2 : \tau}$

because $x \neq x'$

$\Gamma, x:\tau', x':\tau'' \vdash e_2 : \tau$

Case $x = x'$:

We know $\Gamma, x':\tau'' \vdash e_2 : \tau$.

WTS $\Gamma, \underline{x:\tau'}, x':\tau'' \vdash e_2 : \tau$.

$\Gamma, x':\tau'' = \Gamma, x:\tau', x':\tau''$ $\qquad$ $\Gamma, c:\text{Bool} = \begin{Bmatrix} a \mapsto \text{Int} \\ b \mapsto \text{Bool} \\ c \mapsto \text{Bool} \end{Bmatrix}$

$\Gamma = \begin{Bmatrix} a \mapsto \text{Int} \\ b \mapsto \text{Bool} \end{Bmatrix}$ $\qquad$ $\Gamma, c:\text{Int} = \begin{Bmatrix} a \mapsto \text{Int} \\ b \mapsto \text{Bool} \\ c \mapsto \text{Int} \end{Bmatrix}$ $\qquad$ $\Gamma, c:\text{Int}, c:\text{Bool} = \begin{Bmatrix} a \mapsto \text{Int} \\ b \mapsto \text{Bool} \\ c \mapsto \text{Bool} \end{Bmatrix}$

$\Gamma \vdash v : \tau$ and $\Gamma, x:\tau \vdash e : \tau'$ then $\Gamma \vdash e[v/x] : \tau'$

By induction on height of $\Gamma, x:\tau \vdash e : \tau'$ then by case analysis on rule:

Integer case: $e = n$ for $n \in \mathbb{Z}$

$\Gamma, x:\tau \vdash e : \tau'$ is $\dfrac{}{\Gamma, x:\tau \vdash n : \text{Int}}$

Want to show is $\Gamma \vdash n[v/x] : \text{Int}$

$n[v/x] = n$ $\quad \Gamma \vdash n : \text{Int}$ (by Int Rule)

$+$ case: $e = e_1 + e_2$

$\Gamma, x:\tau \vdash e : \tau'$ is $\dfrac{\overbrace{\Gamma, x:\tau \vdash e_1 : \text{Int}} \quad \overbrace{\Gamma, x:\tau \vdash e_2 : \text{Int}}}{\Gamma, x:\tau \vdash e_1 + e_2 : \text{Int}}$

Want to show is

$\Gamma \vdash e[v/x] : \text{Int}$

$\Gamma \vdash (e_1 + e_2)[v/x] : \text{Int}$

$\Gamma \vdash (e_1[v/x]) + (e_2[v/x]) : \text{Int}$

By ind

$\Gamma \vdash e_1[v/x] : \text{Int}$

$\Gamma \vdash e_2[v/x] : \text{Int}$

By + rule

Function Case: $\Gamma, x:\tau \vdash e : \tau'$ is $\dfrac{\Gamma, x:\tau, x':\tau_1 \vdash e' : \tau_2}{\Gamma, x:\tau \vdash (\text{Function } x':\tau_1 \to e') : \tau_1 \to \tau_2}$

WTS $\quad \Gamma \vdash (\text{Function } x':\tau_1 \to e')[v/x] : \tau_1 \to \tau_2$

Case $x \neq x'$: $(\text{Function } x':\tau_1 \to e')[v/x] = \text{Function } x':\tau_1 \to (e'[v/x])$

$\Gamma, x':\tau_1, x:\tau \vdash e' : \tau_2$ by Ind $\quad \Gamma, x':\tau_1 \vdash e'[v/x] : \tau_2$

$\Gamma \vdash \text{Function } x':\tau_1 \to (e'[v/x]) : \tau_1 \to \tau_2$

Case $x = x'$: $(\text{Function } x':\tau_1 \to e')[v/x] = \text{Function } x':\tau_1 \to e'$

$\Gamma \vdash \text{Function } x':\tau_1 \to e' : \tau_1 \to \tau_2$