TFb Soundness: If $e \Rightarrow v$ and $\Gamma \vdash e : \tau$ then $\Gamma \vdash v : \tau$.

- Only talks about expressions that evaluate.
- Induction on expression height doesn't work: expression doesn't necessarily get smaller.

Proof. By induction on the height of the proof of $e \Rightarrow v$.

Base case: height of proof of $e \Rightarrow v$ is 0. The only proof rule w/ height 0 is the Value rule, so $e \Rightarrow v$ must use the Value rule. So $e = v$. Since $\Gamma \vdash e : \tau$ then $\Gamma \vdash v : \tau$. ✓

Inductive step: height of proof of $e \Rightarrow v$ is $k+1$ for $k \in \mathbb{N}$. Proceed by case analysis on the rule used to prove $e \Rightarrow v$.

- If the Plus opsem rule was used then: $e = e_1 + e_2$ for some e_1 and e_2; $e_1 \Rightarrow v_1$ and $e_2 \Rightarrow v_2$ for some v_1 and v_2; $v_1, v_2 \in \mathbb{Z}$; v is the sum of v_1 and v_2.
  By Int type rule, $\Gamma \vdash v_1 : \text{Int}$ and $\Gamma \vdash v_2 : \text{Int}$.
  Because $e = e_1 + e_2$, the proof of $\Gamma \vdash e : \tau$ must use the Plus type rule, so: $\tau = \text{Int}$; $\Gamma \vdash e_1 : \text{Int}$; and $\Gamma \vdash e_2 : \text{Int}$.
  By ind hyp, $\Gamma \vdash v_1 : \text{Int}$ and $\Gamma \vdash v_2 : \text{Int}$.
  Because v is the sum of two integers, $v \in \mathbb{Z}$. By Int rule, $\Gamma \vdash v : \text{Int}$. ✓

- If the If-True rule was used to prove $e \Rightarrow v$ then: $e = \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3$ for some e_1, e_2, e_3; $e_1 \Rightarrow \text{True}$; $e_2 \Rightarrow v$. Because $e = \text{If } e_1 \text{ Then } e_2 \text{ Else } e_3$ then the proof $\Gamma \vdash e : \tau$ must use the If type rule, so: $\Gamma \vdash e_1 : \text{Bool}$; $\Gamma \vdash e_2 : \tau$; $\Gamma \vdash e_3 : \tau$. By ind hyp, $\Gamma \vdash v : \tau$. ✓

- If the Application rule was used to prove $e \Rightarrow v$, $e = e_1\ e_2$ for some e_1, e_2 where: $e_1 \Rightarrow \text{Function } x : \tau' \to e'$; $e_2 \Rightarrow v_2$; $e'[v_2/x] \Rightarrow v$. Because $e = e_1\ e_2$, the proof of $\Gamma \vdash e : \tau$ must use Appl rule, so: $\Gamma \vdash e_1 : \tau' \to \tau$ and $\Gamma \vdash e_2 : \tau'$. By ind hyp, $\Gamma \vdash v_2 : \tau'$. Because e_1 is a function expression, the proof $\Gamma \vdash e_1 : \tau' \to \tau$ must use the Function typing rule, so: $\Gamma\{x : \tau'\} \vdash e' : \tau$. By Substitution Lemma, $\Gamma \vdash e'[v_2/x] : \tau$. By ind hyp, we have $\Gamma \vdash v : \tau$.

$F\flat\ \ddot{\frown}$

$$v ::= \cdots \mid \ddot{\frown}$$

$$\frac{e \Rightarrow \mathit{True}}{\mathit{Not}\ e \Rightarrow \mathit{False}} \qquad \frac{e \Rightarrow v \quad v \in \mathbb{Z}}{\mathit{Not}\ e \Rightarrow \ddot{\frown}}$$

$$\frac{e \Rightarrow \ddot{\frown}}{\mathit{Not}\ e \Rightarrow \ddot{\frown}} \qquad \frac{e_1 \Rightarrow \ddot{\frown}}{e_1 + e_2 \longrightarrow \ddot{\frown}}$$

$$\frac{e_1 \Rightarrow v \quad v \notin \mathbb{Z}}{e_1 + e_2 \Rightarrow \ddot{\frown}}$$

If $e \Rightarrow v$ and $\Gamma \vdash e : \tau$ then $\Gamma \vdash v : \tau$.