TFb Soundness

If $\emptyset \vdash e : \tau$ and $e \Rightarrow v$ then $\emptyset \vdash v : \tau$.

Bad $\overline{\Gamma \vdash e : \text{Int}}$

Lemma: "that which is assumed"
a subproof

TFb Weakening Lemma: If $\Gamma \vdash e : \tau$ and x not free in e then $\Gamma\{x:\tau'\} \vdash e : \tau$.

Proof. By induction on the height of e.

Base case: height of e is 0. Proceed by case analysis on the form of the expression.

- If $e \in \mathbb{Z}$ then by Int rule, $\Gamma\{x:\tau'\} \vdash e : \text{Int}$. The proof of $\Gamma \vdash e : \tau$ must use the Int rule, so $\tau = \text{Int}$. ✓
- If $e \in \{\text{True}, \text{False}\}$ then the proof of $\Gamma \vdash e : \tau$ must use the Boolean rule so $\tau = \text{Bool}$. By Boolean rule, $\Gamma\{x:\tau'\} \vdash e : \text{Bool}$. ✓
- If $e = x'$ for some x' then the proof of $\Gamma \vdash e : \tau$ must use the Variable rule. So $(x':\tau) \in \Gamma$. Because x is not free in e and because x' is free in e, we know $x \neq x'$. Since $x \neq x'$, $(x':\tau) \in \Gamma\{x:\tau'\}$. So, by Variable rule, $\Gamma\{x:\tau'\} \vdash e : \tau$.

So base case is complete.

Inductive step. Assume height of e is $k+1$. Proceed by case analysis on form of e.

- If $e = \text{Not } e'$ for some e' then the proof of $\Gamma \vdash e : \tau$ must use the Not rule. So we have $\Gamma \vdash e' : \text{Bool}$ and that $\tau = \text{Bool}$. By defn of height, e' has height k. By ind hyp. we have $\Gamma\{x:\tau'\} \vdash e' : \text{Bool}$. By Not rule, $\Gamma\{x:\tau'\} \vdash \text{Not } e' : \text{Bool}$. ✓
- etc.

TFb Substitution Lemma: If $\Gamma\{x:\tau_1\} \vdash e:\tau_2$ and $\Gamma \vdash v:\tau_1$ then $\Gamma \vdash e[v/x]:\tau_2$.

Proof. By induction on the height of e.

Base case: e has height 0. Proceed by case analysis on form of e.

- If $e \in \mathbb{Z}$ then the proof of $\Gamma\{x:\tau_1\} \vdash e:\tau_2$ must use the Int rule. So $\tau_2 = \text{Int}$. x is not free in e, so $e[v/x] = e$. It is enough to show $\Gamma \vdash e: \text{Int}$. This is true by the Int rule. ✓
- If $e \in \{\text{True}, \text{False}\}$ then a similar argument applies using the Bool rule.
- If $e = x'$ for some x'. Either $x = x'$ or $x \neq x'$.
  - If $x = x'$ then, as the proof of $\Gamma\{x:\tau_1\} \vdash e:\tau_2$ uses the Var rule, $(x:\tau_1) \in \Gamma\{x:\tau_1\}$ and $\tau_1 = \tau_2$. Because $x = x'$, $e[v/x] = v$. It suffices to show $\Gamma \vdash v:\tau_2$. We know $\Gamma \vdash v:\tau_1$, so we are finished.
  - If $x \neq x'$, then ...