"Lemma" - "given" or "argued"

TFb Weakening Lemma: If $\Gamma \vdash e : \tau$ and x not free in e then $\Gamma, x:\tau' \vdash e : \tau$.

Proof: by induction on the height of e and then by case analysis on the proof rule used in $\Gamma \vdash e:\tau$.

Case of Int Rule: Then $e = n \in \mathbb{Z}$ and $\tau = Int$. Let $\Gamma' = \Gamma, x:\tau'$. Then by the Int rule, $\Gamma' \vdash n : Int$.

Case of Plus Rule: Then $e = e_1 + e_2$ and $\tau = Int$. Also, we know $\Gamma \vdash e_1 : Int$ and $\Gamma \vdash e_2 : Int$ from premises of Plus Rule. By inductive hypothesis, $\Gamma, x:\tau' \vdash e_1 : Int$ and $\Gamma, x:\tau' \vdash e_2 : Int$. By Plus Rule $\Gamma, x:\tau' \vdash e : Int$.

Case of Let Rule: Then $e = (\text{Let } x':\tau'' = e_1 \text{ In } e_2)$. Also, we know $\Gamma \vdash e_1 : \tau''$ and $\Gamma, x':\tau'' \vdash e_2 : \tau$. Let $\Gamma' = \Gamma, x:\tau'$. By inductive hypothesis, $\Gamma' \vdash e_1 : \tau''$. We have two cases: either $x = x'$ or $x \neq x'$.

If $x \neq x'$ then x not free in e_2. By inductive hypothesis, $\Gamma, x':\tau'', x:\tau' \vdash e_2 : \tau$. WTS $\Gamma', x':\tau'' \vdash e_2 : \tau$. WTS $\Gamma, x:\tau', x':\tau'' \vdash e_2 : \tau$. Since $x \neq x'$, $\Gamma, x:\tau', x':\tau'' = \Gamma, x':\tau'', x:\tau'$. So we know $\Gamma', x':\tau'' \vdash e_2 : \tau$.

$\Gamma = \{a \mapsto Bool,\ b \mapsto Int\}$

$\Gamma, c:Bool = \{a \mapsto Bool,\ b \mapsto Int,\ c \mapsto Bool\}$

$\Gamma, c:Bool, c:Int = \{a \mapsto Bool,\ b \mapsto Int,\ c \mapsto Int\}$

By Let rule, $\Gamma' \vdash e : \tau$.

Otherwise, $x = x'$. Then $\overset{\Gamma'}{\boxed{\Gamma, x:\tau'}}, x':\tau'' = \Gamma, x':\tau''$. Since we know that $\Gamma, x':\tau'' \vdash e_2 : \tau$, we know $\Gamma', x':\tau'' \vdash e_2 : \tau$. By Let rule, $\Gamma' \vdash e : \tau$.

etc. etc.

TFb Substitution Lemma: If $\Gamma, x:\tau \vdash e : \tau'$ and $\Gamma \vdash v : \tau$ then $\Gamma \vdash e[v/x] : \tau'$.

By induction on height of e and by case analysis on the rule used. Let $\Gamma' = \Gamma, x:\tau$.

Case of Int Rule: Then $e = n \in \mathbb{Z}$ and $\tau' = Int$. $n[v/x] = n$. By Int rule, $\Gamma \vdash e[v/x] : Int$.

Case of Plus Rule: Then $e = e_1 + e_2$ and $\tau' = Int$. Also $\Gamma' \vdash e_1 : Int$ and $\Gamma' \vdash e_2 : Int$ by premises. $e[v/x] = (e_1 + e_2)[v/x] = (e_1[v/x]) + (e_2[v/x])$. By inductive hyp., $\Gamma \vdash e_1[v/x] : Int$ and $\Gamma \vdash e_2[v/x] : Int$. So by Plus rule, $\Gamma \vdash (e_1[v/x]) + (e_2[v/x]) : Int$ and so $\Gamma \vdash e[v/x] : Int$.

$$\text{Plus}\ \frac{\Gamma \vdash e_1 : Int \qquad \Gamma \vdash e_2 : Int}{\Gamma \vdash e_1 + e_2 : Int}$$