CS97: Week 2 Lab

Due 1:00am 09/21/2015
This lab is worth having some fun with (and 50 points!). Submit your lab via email to sindhu@cs.swarthmore.edu with the subject line CS97: Week 2 Lab

1 Probability Warm-up (10 points)

Let $S$ be the sample space and $E \subseteq S$ be an event in the sample space. For example, $S = \{H, T\}$ would be the sample space for the outcome of a coin toss. Here $H$ corresponds to an outcome of heads and $T$ to an outcome of tails. A valid probability distribution $P$ satisfies the following axioms $^1$

- $\forall E \subseteq S$, $0 \leq P(E) \leq 1$
- $P(S) = 1$
- For any mutually exclusive events $E_1, \ldots, E_n$ ($E_i \cap E_j = \emptyset$ for $i \neq j$),
  \[ P(E_1 \cup \ldots \cup E_n) = \sum_{i=1}^{n} P(E_i) \]

Suppose we toss two fair coins and record their faces.

1. (1 point) In set notation, write out the sample space of this experiment.
2. (2 point) In set notation, write out the event that both coins come up heads.
3. (2 points) Assuming that every outcome in the sample space is equally likely, what is the probability of the event that both coins come up heads?

$^1$ this is (a) informal (b) in set theoretic notation and (c) for a finite sample space
4. (3 points) Let \( E_1 \) be the event that the first coin comes up heads and \( E_2 \) be the event that the second coin comes up tails. Are these events mutually exclusive? If so, show that the third axiom of probability holds. If not, compute the left hand side and the right hand side of the axiom’s statement. Again, you may assume that every outcome of the sample space is equally likely.

5. (2 points) Let \( X \) be a random variable that takes value 1 when the faces of the two coins match and 0 otherwise. What is the expected value of \( X \), if every outcome of the sample space is equally likely? Be explicit about the formula you are using here!

2 Swarthmore Labor Day Game (20 points)

To make it fun to be in school (and in lab) on Labor Day (especially while everyone else is out grilling) you and I are going to play a game. We will use a biased coin that comes up heads two-thirds of the time. Here are the rules of the game:

- I am going to toss this coin repeatedly until it comes up tails.
- I will pay you $3^n$ for \( n \) tosses (a sequence of \( n - 1 \) heads and a tail).

Obviously, if I am willing to pay you something at the end of the game, I would want to know what you are willing to pay me to participate. We are going to figure out the price of entry you should be willing to pay for this game (based solely in probability theory).

1. (5 points) What is the probability that I will toss the coin exactly \( n \) times?
2. (10 points) What is the amount you can expect to win in this game?
3. (5 points) Up to what amount would you be willing to pay so that you have a non-negative expected payoff? Does this seem realistic?
3 Risk-neutral Trading (20 points)

In this problem we will analyze the behavior of a risk-neutral trader. For our purposes, a risk-neutral trader can be treated as an entity that makes purchases that have a non-negative expected payoff.

Suppose you believe that there is a probability $p$ that will rain tomorrow. Consider a contract that pays off $1 if it rains tomorrow and nothing otherwise.

1. (5 points) I claim that there exists a maximum amount $x$ above which no ‘rational’ risk-neutral trader would be willing to pay for this contract. Am I right? If so, what is $x$? If not, what is the flaw in my claim?

2. (10 points) If you purchase this contract at price $q$, what is your expected payoff?

3. (5 points) Quantify the relationship between $p$ and $q$ for which you would be willing to make this purchase.