Instructions:

- Please write your name and username in the space provided.
- You have the entirety of lab time to finish this exam.
- This is a closed book, closed notes exam. However, you may have access to a “reminder” sheet restricted to one side of a US Letter size paper (8.5" × 11").
- Please turn off all cell phones and electronic devices. Headphones, computers, wireless devices, notebooks, textbooks, and other external objects that are or appear to be capable of communication, computation, or storage must be stowed out of reach during the exam.
- Please write your solutions in the space provided. If you run out of room for an answer, continue on the back of the page.
  
  - Provide a brief description of any machines you construct in your proofs.
  - Write legibly! In extreme cases, lack of readability may result in a points penalty.
  - Unless otherwise instructed, show enough work to justify your answer.
  - When giving proofs, you may use any results (including closure properties, constructions, and theorems) shown in class, in the text, in the practice problems or homework without proving those results.
  - You should attempt all problems. Partial solutions will receive partial credit.
  
- Good luck!
Variants of TMs

1. TMs with stay-put
2. Multitape TMs
3. Nondeterministic TMs (NTMs)
4. TM with output (enumerators)
Proving Equivalence of TM variants

Show that the class of languages recognized by multitape TMs is the same as the class of languages recognized by single tape TMs.
given: single tape TM
construct: k-tape TM

ignore read/write heads and tapes #2-k
given: k-tape TM
construct: single tape TM

specify:
• representation of starting configuration
• representation of intermediate configuration
• any transition from intermediate configuration
starting configuration

```
M
```

```
\[
\begin{array}{ccccccc}
  a & a & b & u & u & u & u & u & u & u & \ldots \\
  u & u & u & u & u & u & u & u & u & u & \ldots \\
  u & u & u & u & u & u & u & u & u & u & \ldots \\
  \# & a & a & b & \# & u & \# & u & \# & u & \ldots \\
\end{array}
\]
```
intermediate configuration

M

S
1. remember current (marked) symbols
2. replace marked symbols as indicated by M’s transition function
3. move marked symbols
   a. possibly shift tape contents right
Notion of Encoding

\[ A_{DFA} = \{ \langle D, w \rangle : D \text{ is a DFA that accepts input } w \} \]
Decidable Languages

$A_{DFA} = \{ \langle D, w \rangle : D \text{ is a DFA that accepts input } w \}$
EQ\textsubscript{DFA} = \{ \langle D_1, D_2 \rangle : D_1, D_2 \text{ are DFAs} \\
\quad \quad \text{and } L(D_1) = L(D_2) \}\n
\textbf{idea: use emptiness testing of the symmetric difference of } L(D_1) \text{ and } L(D_2)\n
L(C) = \left( L(A) \cap \overline{L(B)} \right) \cup \left( \overline{L(A)} \cap L(B) \right)
Prove that $\text{EQ}_{\text{DFA}}$ is decidable

$\text{EQ}_{\text{DFA}} = \{ \langle D_1, D_2 \rangle : D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2) \}$

Let $D$ be the DFA st $L(D) = \text{SymmDiff}(L(D_1), L(D_2))$

**Observe:**
1. Given $D_1$ and $D_2$, $D$ can be constructed because of closure of regular languages under union, intersection and complement
2. $\langle D_1, D_2 \rangle \in \text{EQ}_{\text{DFA}} \iff D \in \text{E}_{\text{DFA}}$

$D_{\text{EQ}}$

On input $\langle D_1, D_2 \rangle$:
- Construct DFA $D$ such that $L(D)$ is the symmetric difference of $L(D_1)$ and $L(D_2)$
- Run the decider $D_E$ (for $\text{E}_{\text{DFA}}$) on DFA $D$
  - accept if $D_E$ accepts
  - reject if $D_E$ rejects
Proving Turing Recognizability

$A_{TM}$

Enumerable languages
A language is Turing recognizable iff some enumerator enumerates it
Proving Undecidability

Typically reduction from some known undecidable language
Prove that $E_{\text{TM}}$ is undecidable

$E_{\text{TM}} = \{ \langle M \rangle : M \text{ is a TM and } L(M) = \emptyset \}$

Proof by contradiction. Suppose $D_E$ decides $E_{\text{TM}}$. Construct $D_A$

On input $\langle M, w \rangle$:

- Construct $N_{M,w}$:
  - On input $x$
    - if $x \neq w$, reject.
    - if $x = w$, Simulate $M$ on $w$ and accept if $M$ accepts.
- Run the decider $D_E$ (for $E_{\text{TM}}$) on $\langle N_{M,w} \rangle$
  - reject if $D_E$ accepts
  - accept if $D_E$ rejects

Observe:
1. If $M$ accepts $w$, $L(N_{M,w}) = \{w\}$
2. If $M$ does not accept $w$, $L(N_{M,w}) = \{}$
3. $\langle N_{M,w} \rangle \in E_{\text{TM}} \leftrightarrow \langle M, w \rangle \in A_{\text{TM}}$
4. Thus, $D_A$ decides $A_{\text{TM}}$ which we know is undecidable.
Contradiction.

A language is **decidable** iff it is **Turing-recognizable** and **co-Turing-recognizable**.
Countable sets
and the Diagonalization method
All languages

Turing recognizable

CFL

Turing decidable

RL
True or False

Every undecidable language is infinite.

True
True or False

A = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is uncountable}\} is undecidable.

False
True or False

If $L_1$ is regular and $L_2$ is decidable, then $L_1 \cap L_2$ is decidable.

True