Prediction Markets, Exponential Families and Graphical Models

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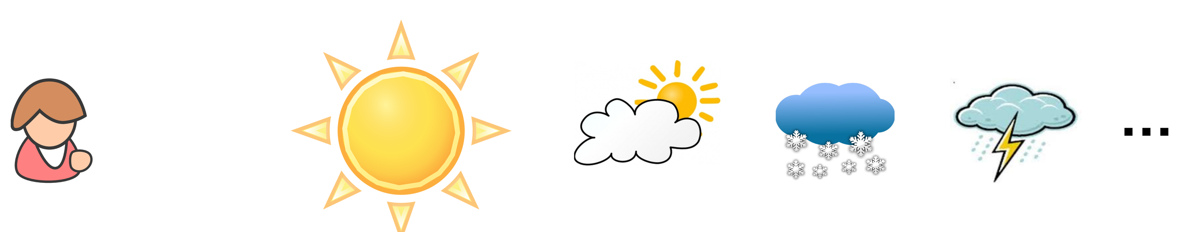
We design a class of proper scoring rules for infinite outcome spaces based on maximum entropy distributions that elicit statistics of the data. We define a cost function based prediction market based on this scoring rule and consider the semantics of information aggregation in this market. We also explicitly characterize the interaction between markets on related events by drawing on results from graphical models. In one instance, we are able to show that a trader in such a market behaves as if he were implementing a learning algorithm, even though his incentives are purely financial.

Generalized LMSR



Idea: Elicit *summary information* f(x)
e.g., f(x) = (x, x^2)

agent believes $m = E_r[f(x)]$ for any belief distribution r

Score is log p(x; m) for the maxent distribution p

$\max_p \; -\int p(x) \log p(x)\, dx \quad \text{s.t.} \; E_p[f(x)] = m$

how to elicit beliefs on
large (**infinite**) outcome spaces

Exponential Family Prediction Market

Exponential Families	Cost Function Prediction Market
sufficient statistics	contract payoff
log partition function	cost function
natural parameters	outstanding share vector
mean parameter	market prices

$$P_\beta(x) = e^{\boldsymbol{\beta}\cdot\boldsymbol{\phi}(x) - \psi(\boldsymbol{\beta})}$$

$$\psi(\boldsymbol{\beta}) = \log \int e^{\boldsymbol{\beta}\cdot\boldsymbol{\phi}(x)} dx \qquad C(\mathbf{q}) = \log \int e^{\mathbf{q}^T \Phi(x)}\, dx$$

$$\nabla C(\mathbf{q}) = E_\beta[\Phi(x)]$$

continuous outcome space

Gaussian Market

outcome drawn from a normal distribution with
unknown mean and variance

Cost Function $\frac{\mu^2}{2\sigma^2} + \ln|\sigma|$

share vector $\begin{pmatrix} \frac{\mu}{\sigma^2} \\ \frac{-1}{2\sigma^2} \end{pmatrix}$

2 securities

contract payoff

x	x^2

current market state $\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$ → final market state $\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$

purchase $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

μ=1
σ^2=1

Cost is **C([1, -1/2]ᵀ) - C([0, -1/2]ᵀ)** = $1
Payoff is **1.x + 0.x²** = $x

Bayesian Traders

$\nabla C(\theta) = \nu$

θ → risk neutral → $\theta+\delta$

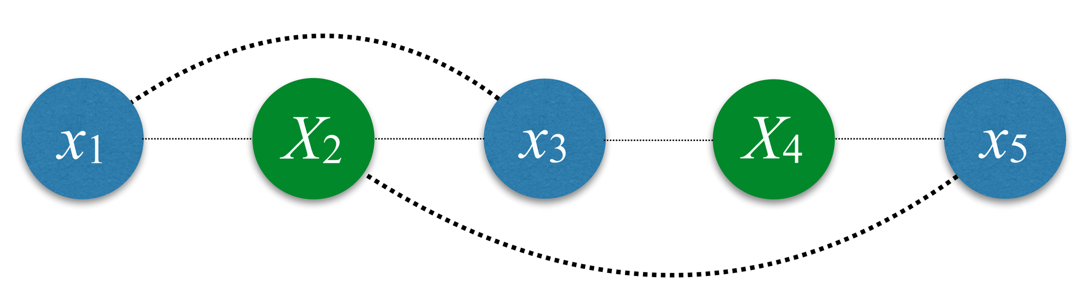
$\nabla C(\theta+\delta) = \frac{n\nu + m\hat{\mu}}{n+m}$

$\hat{\mu} = \frac{\sum_{i=1}^m x_i}{m}$

$$\nabla C(\theta+\delta) = \frac{n\nu + \sum_i m_i \hat{\mu}_i}{n + \sum_i m_i}$$

final market *prices*
average of **initial** market prices and
mean parameter **beliefs** of all agents

Partially informed traders



- Current market estimate p_θ
- Optimal trade for a trader with partial data

$$\arg\max_{\theta'} \{(\theta' - \theta)\cdot \mathbb{E}_{p_\theta}[\phi(x_1, X_2, x_3, X_4, x_5)] - (C(\theta') - C(\theta))\}$$

EM algorithm

E step $\quad \mu^{t+1} \leftarrow \mathbb{E}_{p_\theta}[\phi(x_1, X_2, x_3, X_4, x_5)]$

M step $\quad \theta^{t+1} \leftarrow \arg\max_\theta \{\mu^{t+1}\cdot\theta - C(\theta)\}$

Joint Markets

- Joint distribution GMRF p_θ
- Payoff of contracts are x_i, x_i^2, $x_i x_{i+1}$

$$p(\mathbf{x}) \propto \mathbb{E}\left\{\sum_{i=1}^5 \theta_i x_i + \frac{1}{2}\sum_{i=1}^5 \Theta_{ii} x_i^2 + \sum_{i=1}^4 \Theta_{i,i+1} x_i x_{i+1}\right\}$$

Optimal trade if interested only in x_1

$E[x_1] = m$, $E[x_1^2] = m^2 + v^2$

$\nabla_{\theta_1} C(\theta, \Theta) = m$
$\nabla_{\Theta_{11}} C(\theta, \Theta) = m^2 + v^2$
→ $\min_{\theta_1, \Theta_{11}} \{C(\theta_1, \Theta_{11}) - m\theta_1 - (m^2+v^2)\Theta_{11}\}$

Effect on p(x_3) for purchase of δ shares in θ_1

$$\mu_3' = \mu_3 + \sigma_{31}\delta \qquad \Sigma_{ij}' = \Sigma_{ij} - \frac{\delta\sigma_{1i}\sigma_{1j}}{1+\delta\sigma_{11}}$$

Risk Averse Traders

potential games-like argument $\quad U_a(w) = -\frac{1}{a}\exp(-aw)$

$$U_i(\vec{\delta}) - U_i(\vec{\delta}_{-i}, \delta_i') = F(\vec{\delta}) - F(\vec{\delta}_{-i}, \delta_i')$$

Nash Equilibrium ⇔ local optimum

$$F(\vec{\delta}) = \psi\left(\theta + \sum_i \delta_i\right) + \sum_i \frac{1}{a_i}\psi(\hat{\theta}_i - a_i\delta_i)$$

$$\theta + \sum_{j=1}^n \delta_j = \frac{\theta + \sum_{i=1}^n \left(\frac{\hat{\theta}_i}{a_i}\right)}{1 + \sum_{i=1}^n \frac{1}{a_i}}$$

equilibrium state is
a convex combination of
current market state and
weighted agent ***beliefs***