CS 31: Intro to Systems Binary Arithmetic

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Reading Quiz

Unsigned Integers

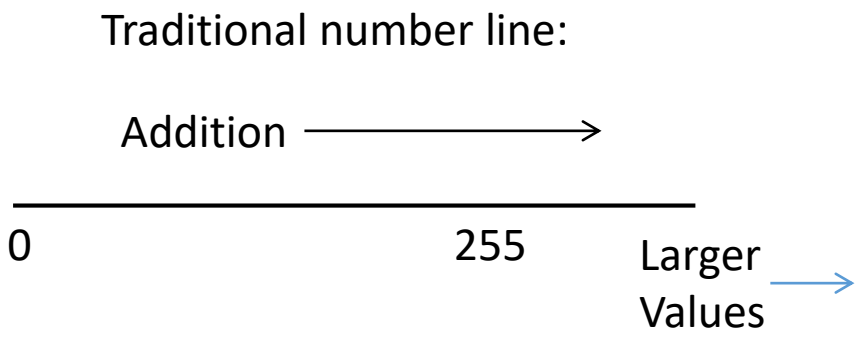
- Suppose we had one byte
  - Can represent 2^8 (256) values
  - If unsigned (strictly non-negative): 0 – 255

252 = 11111100

253 = 11111101

254 = 11111110

255 = 11111111



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Car odometer “rolls over”.

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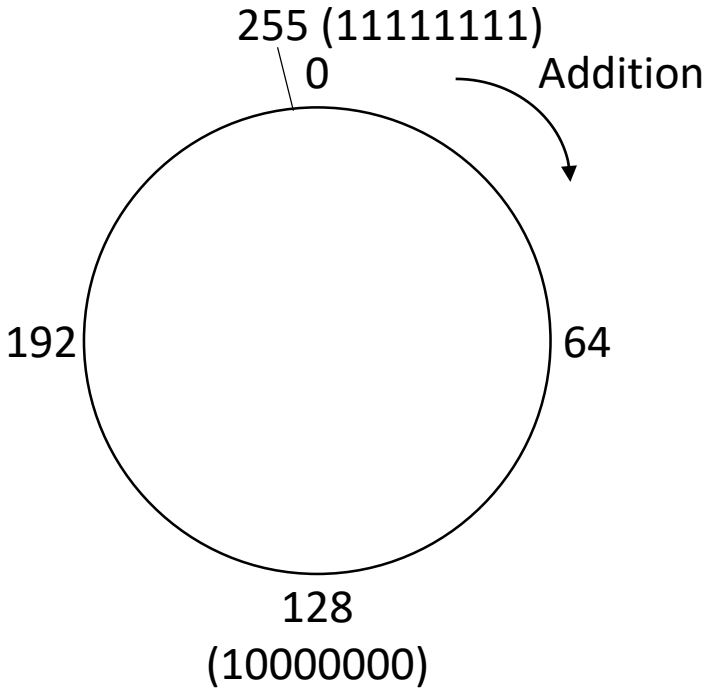


Modular arithmetic: Here, all values are modulo 256.

Unsigned Addition (4-bit)

- Addition works like grade school addition:

```
   1
   0110      6
+  0100   +  4
------    ----
   1010     10
```

Four bits give us range: 0 - 15

Unsigned Addition (4-bit)

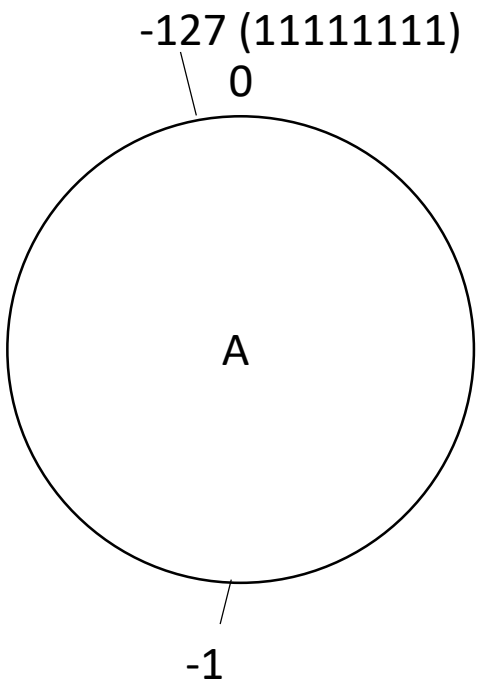
- Addition works like grade school addition:

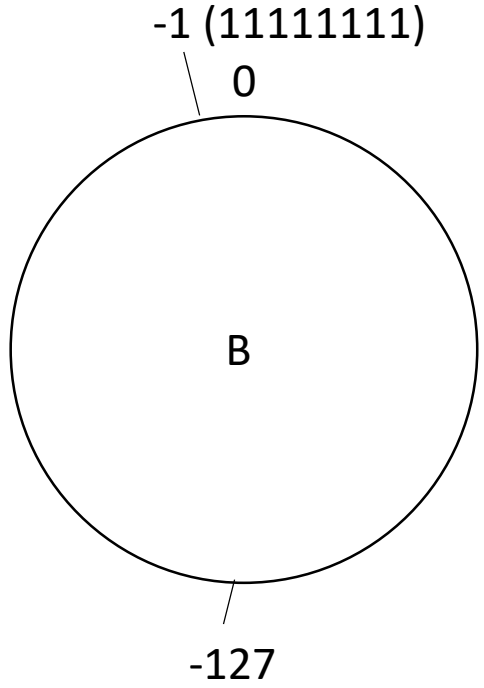
```
   1
  0110      6          1100     12
+ 0100    + 4        + 1010    +10
------    ---        ------    ---
  1010     10        1 0110      6
                     ^carry out
```

Overflow!

Four bits give us range: 0 - 15

Suppose we want to support signed values too (positive **and** negative). Where should we put -1 and -127 on the circle? Why?





C: Put them somewhere else.

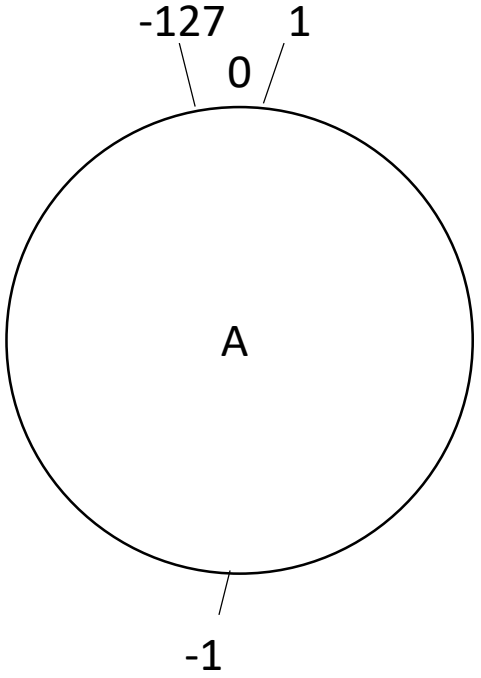
Signed Magnitude

- One bit (usually left-most) signals:
  - 0 for positive
  - 1 for negative

For one byte:

1 = 00000001, -1 = 10000001

Pros: Negation is very simple!



Signed Magnitude

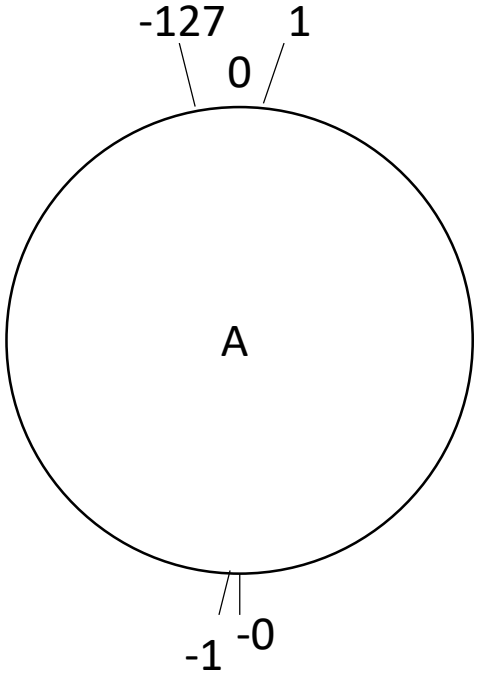
- One bit (usually left-most) signals:
  - 0 for positive
  - 1 for negative

For one byte:

0 = 00000000

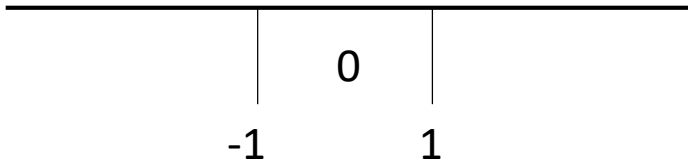
What about 10000000?

Major con: Two ways to represent zero.



Two’s Complement (signed)

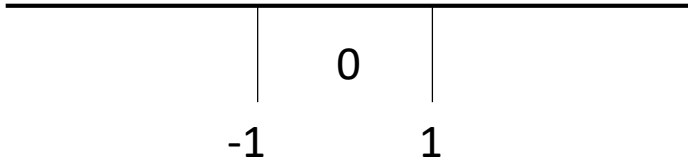
- Borrow nice property from number line:



Only one instance of zero!
Implies: -1 and 1 on either side of it.

Two’s Complement

- Borrow nice property from number line:



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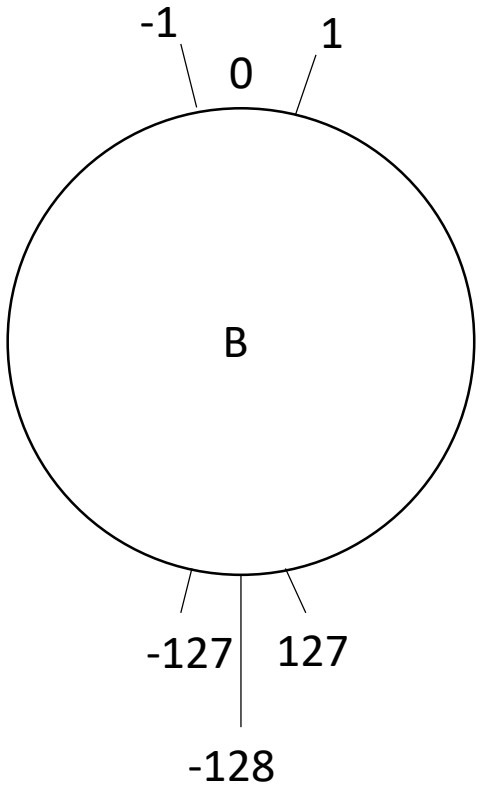


Two's Complement

- Only one value for zero
- With N bits, can represent the range:
  - -2^{N-1} to $2^{N-1} - 1$
- First bit still designates positive (0) /negative (1)

- Negating a value is slightly more complicated:

  1 = 00000001, -1 = 11111111

From now on, unless we explicitly say otherwise, we'll assume all integers are stored using two's complement! This is the standard!

Two's Compliment

- Each two's compliment number is now:

$$[-2^{n-1}*d_{n-1}] + [2^{n-2}*d_{n-2}] + \ldots + [2^{1}*d_{1}] + [2^{0}*d_{0}]$$

Note the negative sign on just the first digit. This is why first digit tells us negative vs. positive.

If we interpret 11001 as a two's complement number, what is the value in decimal?

- Each two's compliment number is now:

$$[-2^{n-1}*d_{n-1}] + [2^{n-2}*d_{n-2}] + \ldots + [2^{1}*d_{1}] + [2^{0}*d_{0}]$$

A. -2

B. -7

C. -9

D. -25

“If we interpret…”

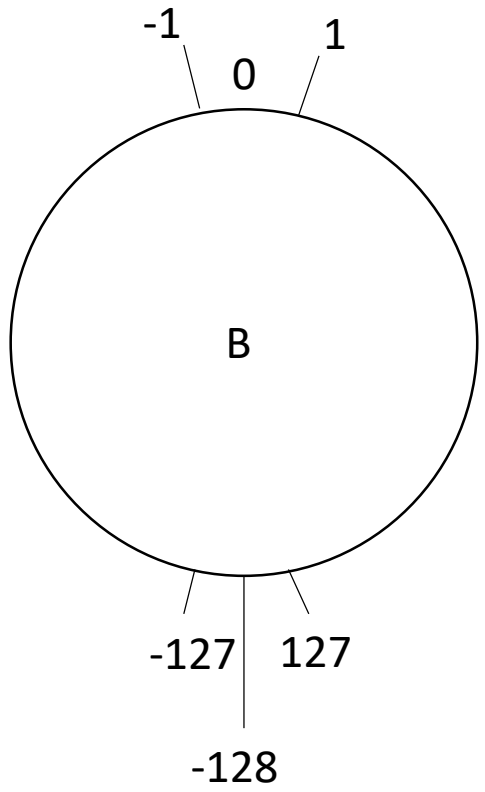
- What is the decimal value of 1100?

- …as unsigned, 4-bit value: 12 (%u)
- …as signed (two’s comp), 4-bit value: -4 (%d)

- …as an 8-bit value: 12
  (i.e., **0000**1100)

Two's Complement Negation

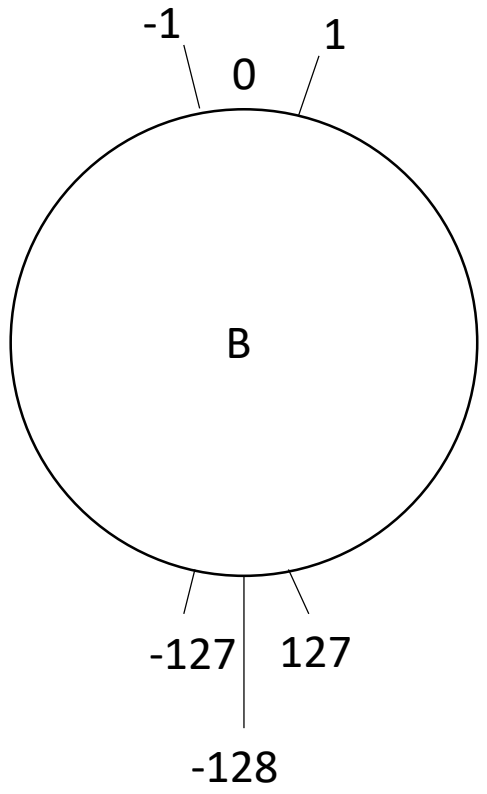
- To negate a value x, we want to find y such that x + y = 0.
- For N bits, $y = 2^N - x$



Negation Example (8 bits)

- For N bits, $y = 2^N - x$
- Negate 00000010 (2)
  - $2^8 - 2 = 256 - 2 = 254$

- Our wheel only goes to 127!
  - Put -2 where 254 would be if wheel was unsigned.
  - 254 in binary is 11111110



Given 11111110, it's 254 if interpreted as unsigned and -2 interpreted as signed.

Negation Shortcut

- A much easier, faster way to negate:
  - Flip the bits (0's become 1's, 1's become 0's)
  - Add 1

- Negate 00101110 (46)
  - 2^8 - 46 = 256 - 46 = 210
  - 210 in binary is 11010010

Addition & Subtraction

- Addition is the same as for unsigned
  - One exception: different rules for overflow
  - Can use the same hardware for both

- Subtraction is the same operation as addition
  - Just need to negate the second operand…

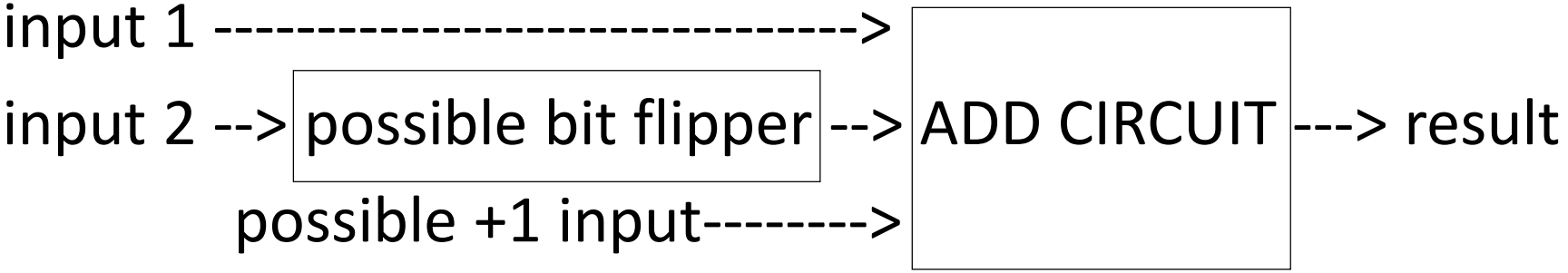
- 6 - 7 = 6 + (-7) = 6 + (~7 + 1)
  - ~7 is shorthand for “flip the bits of 7”

Subtraction Hardware

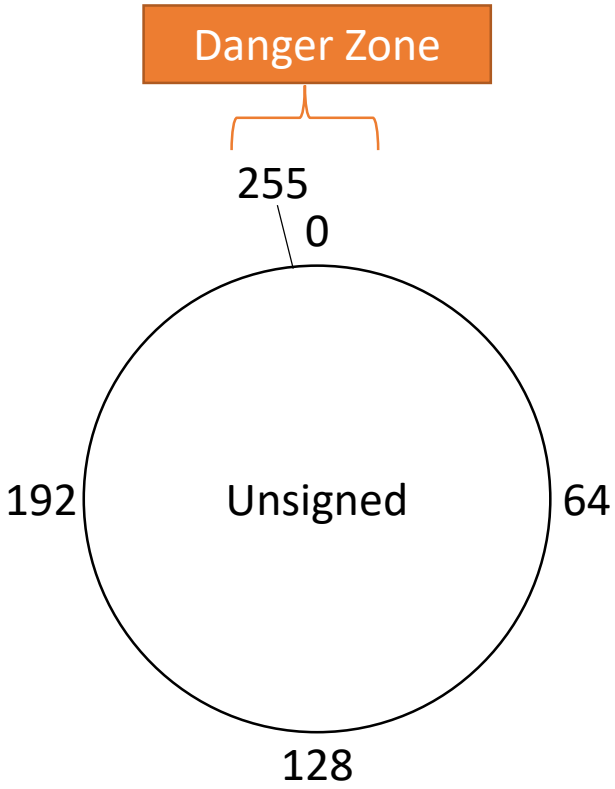
<u>Negate and add 1 to second operand:</u>

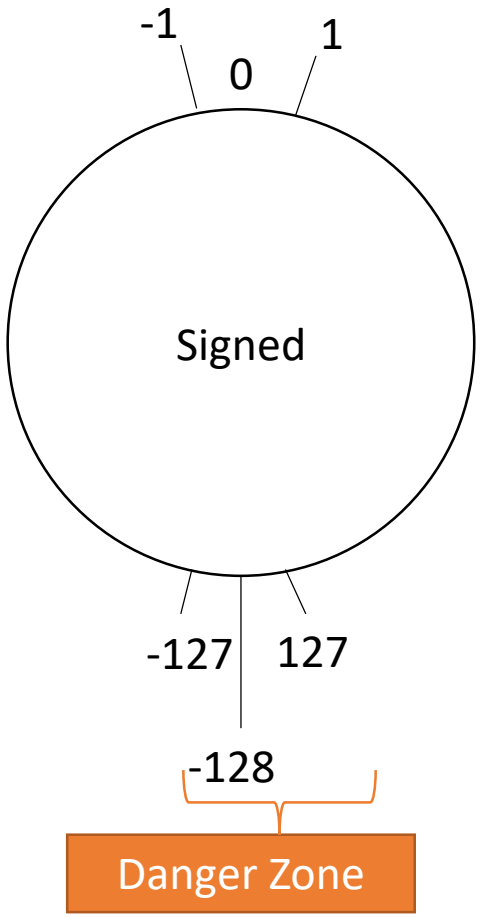
Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1



Overflow, Revisited



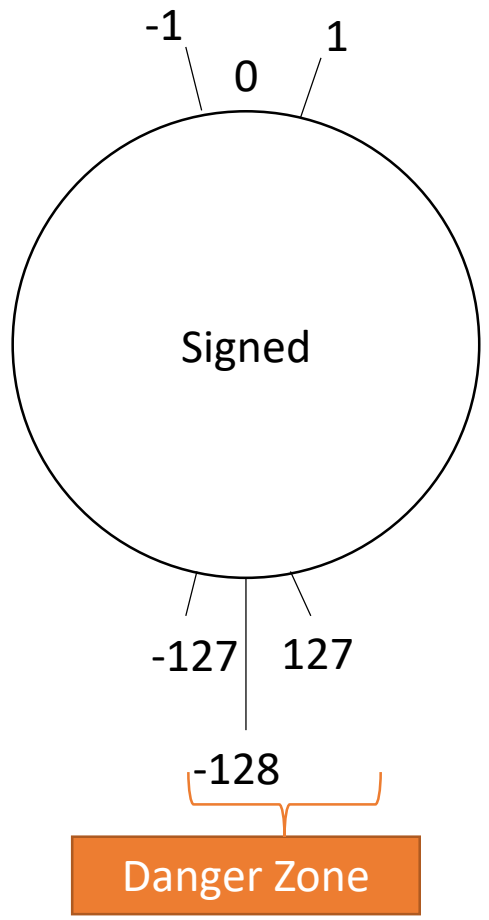


If we add a positive number and a negative number, will we have overflow? (Assume they are the same # of bits)

A. Always

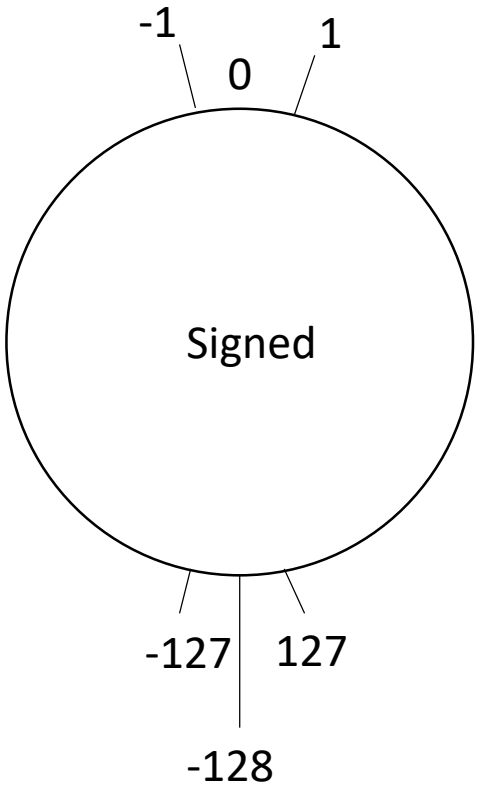
B. Sometimes

C. Never



Signed Overflow

- Overflow: IFF the sign bits of operands are the same, but the sign bit of result is different.
  - Not enough bits to store result!



Signed addition (and subtraction):

2+-1=1	2+-2=0	2+-4=-2
0010	**0**010	**0**010
+**1**111	+**1**110	+**1**100
1 0001	1 0000	1110

No chance of overflow here - signs of operands are different!

Signed Overflow

- Overflow: IFF the sign bits of operands are the same, but the sign bit of result is different.
  - Not enough bits to store result!

Signed addition (and subtraction):

2+-1=1	2+-2=0	2+-4=-2	2+7=-7	-2+-7=7
0010	0010	0010	**0**010	**1**110
+1111	+1110	+1100	+**0**111	+**1**001
1 0001	1 0000	1110	**1**001	1 **0**111

Overflow here! Operand signs are the same, and they don't match output sign!

Overflow Rules

- Signed:
  - The sign bits of operands are the same, but the sign bit of result is different.

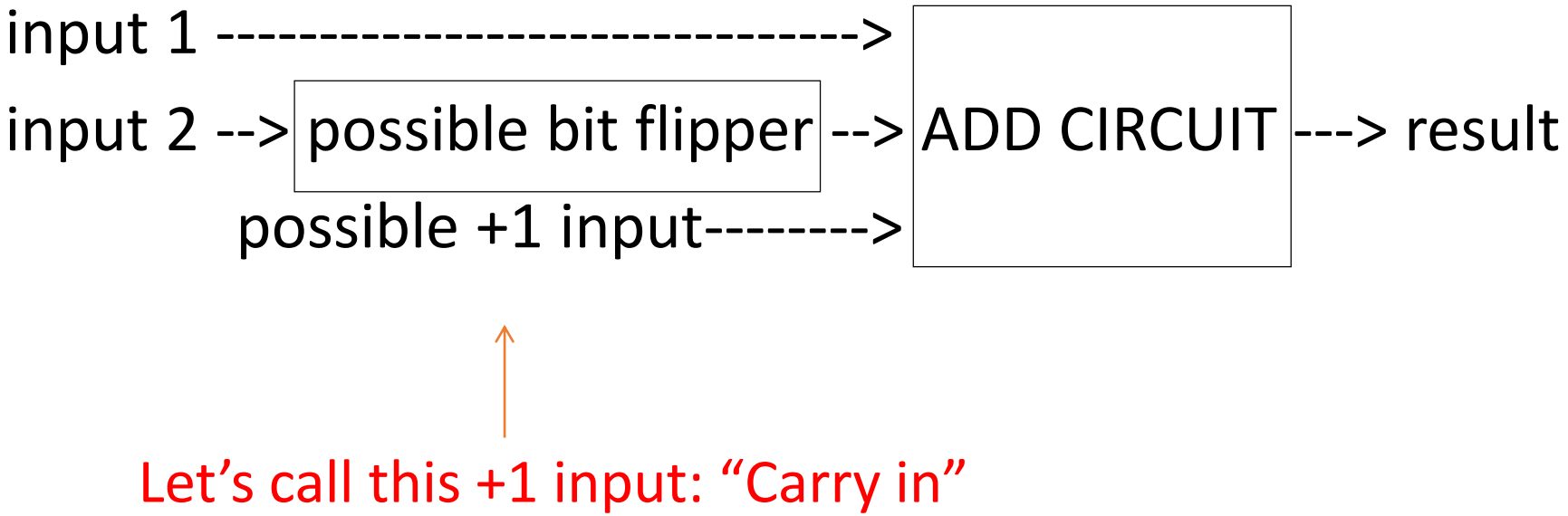
- Can we formalize unsigned overflow?
  - Need to include subtraction too, skipped it before.

Recall Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1



Let’s call this +1 input: “Carry in”

How many of these <u>unsigned</u> operations have overflowed?

4 bit unsigned values (range 0 to 15):

Addition (carry-in = 0)

```
                                        carry-in   carry-out
                                           ↓          ↓
  9 + 11  =   1001 + 1011 + 0 =   1   0100
  9 +  6  =   1001 + 0110 + 0 =   0   1111
  3 +  6  =   0011 + 0110 + 0 =   0   1001
```

Subtraction (carry-in = 1)

```
                                        (-3)
                                      ┌──────┐
  6 -  3  =   0110 + 1100 + 1   = 1   0011
  3 -  6  =   0011 + 1001 + 1   = 0   1101
                   └──────┘
                    (-6)
```

A. 1
B. 2
C. 3
D. 4
E. 5

How many of these unsigned operations have overflowed?

Interpret these as 4-bit unsigned values (range 0 to 15):

Addition (carry-in = 0)

```
                                        carry-in   carry-out
                                           ↓          ↓
  9 + 11  =   1001 + 1011 + 0 =   1   0100 =  4
  9 +  6  =   1001 + 0110 + 0 =   0   1111 = 15
  3 +  6  =   0011 + 0110 + 0 =   0   1001 =  9
```

Subtraction (carry-in = 1)

```
                                        (-3)
                                      ┌─────┐
  6 -  3  =   0110 + 1100 + 1  = 1   0011 =  3
  3 -  6  =   0011 + 1001 + 1  = 0   1101 = 13
                                      └─────┘
                                        (-6)
```

Pattern?

A. 1
B. 2
C. 3
D. 4
E. 5

Overflow Rule Summary

- Signed overflow:
  - The sign bits of operands are the same, but the sign bit of result is different.

- Unsigned: overflow
  - The carry-in bit is different from the carry-out.

C_{in}	C_{out}	C_{in} XOR C_{out}
0	0	0
0	**1**	**1**
1	**0**	**1**
1	1	0

So far, all arithmetic on values that were the same size. What if they're different?

Sign Extension

- When combining signed values of different sizes, expand the smaller value to equivalent larger size:

```
char y=2, x=-13;
short z = 10;
```

```
   z = z + y;
```

```
0000000000001010
+       00000010
0000000000000010
```

```
   z = z + x;
```

```
0000000000000101
+       11110011
1111111111110011
```

Fill in high-order bits with sign-bit value to get same numeric value in larger number of bytes.

Let’s verify that this works

4-bit signed value, sign extend to 8-bits, is it the same value?

0111 ---> 0000 0111 obviously still 7

1010 ----> 1111 1010 is this still -6?

-128 + 64 + 32 + 16 + 8 + 0 + 2 + 0 = -6 yes!

Operations on Bits

- For these, doesn’t matter how the bits are interpreted (signed vs. unsigned)
- Bit-wise operators (AND, OR, NOT, XOR)
- Bit shifting

Bit-wise Operators

- bit operands, bit result (interpret as you please)

& (AND) | (OR) ~(NOT) ^(XOR)

A	B	A & B	A | B	~A	A ^ B
0	0	0	0	1	0
0	1	0	1	1	1
1	0	0	1	0	1
1	1	1	1	0	0

```
  01010101      01101010      10101010     ~10101111
| 00100001    & 10111011    ^ 01101001     ---------
  --------    ----------    ----------     01010000
  01110101      00101010      11000011
```

More Operations on Bits

- Bit-shift operators: << left shift, >> right shift

```
01010101 << 2  is 01010100
                  2 high-order bits shifted out
                  2 low-order bits filled with 0
01101010 << 4  is 10100000
01010101 >> 2  is 00010101
01101010 >> 4  is 00000110

10101100 >> 2  is 00101011 (logical shift)
               or 11101011 (arithmetic shift)
```

Arithmetic right shift: fills high-order bits w/sign bit

C automatically decides which to use based on type:

signed: arithmetic, unsigned: logical

Up Next

- Circuits