CS 31: Intro to Systems
Binary Arithmetic

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Reading Quiz

Unsigned Integers

- Suppose we had one byte
  - Can represent 2^8 (256) values
  - If unsigned (strictly non-negative): 0 – 255

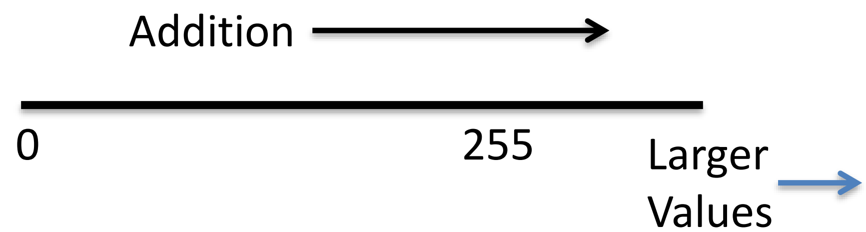
252 = 11111100

253 = 11111101

254 = 11111110

255 = 11111111

Traditional number line:



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What if we add one more?

Car odometer “rolls over”.

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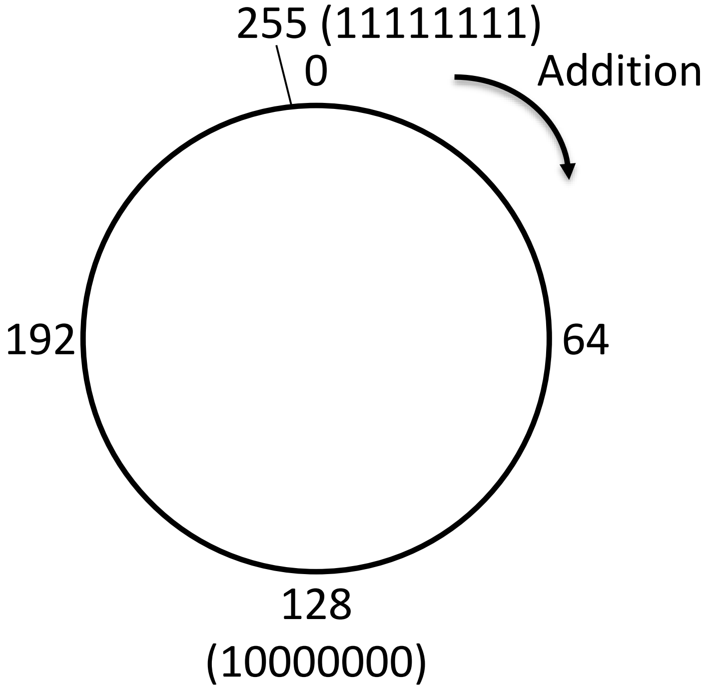


Modular arithmetic: Here, all values are modulo 256.

Unsigned Addition (4-bit)

- Addition works like grade school addition:

```
     1
     0110       6
  +  0100    +  4
  -------    ----
     1010      10
```

Four bits give us range: 0 - 15

Unsigned Addition (4-bit)

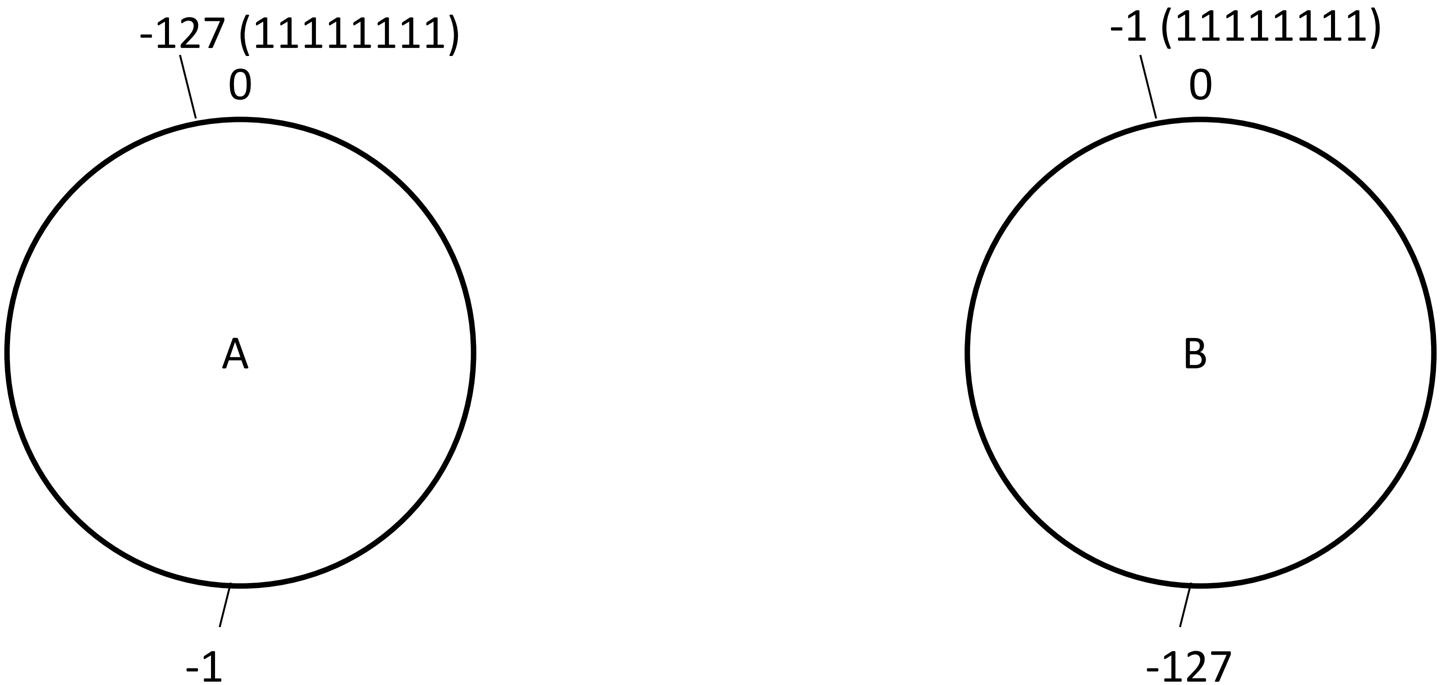
- Addition works like grade school addition:

```
    1
    0110      6        1100     12
  + 0100    + 4      + 1010    +10
  ------    ---      ------    ---
    1010     10      1 0110      6
                     ^carry out
```

Four bits give us range: 0 - 15

Overflow!

Suppose we want to support signed values too (positive **and** negative). Where should we put -1 and -127 on the circle? Why?



C: Put them somewhere else.

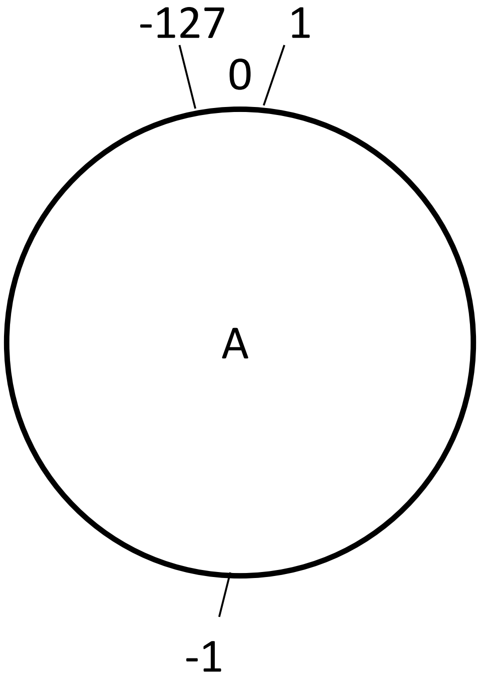
Signed Magnitude

- One bit (usually left-most) signals:
  - 0 for positive
  - 1 for negative

For one byte:

1 = 00000001,   -1 = 10000001

Pros: Negation is very simple!



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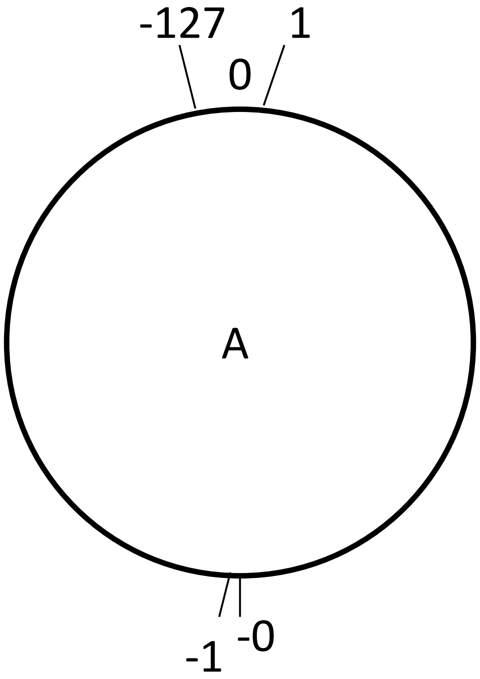


For one byte:

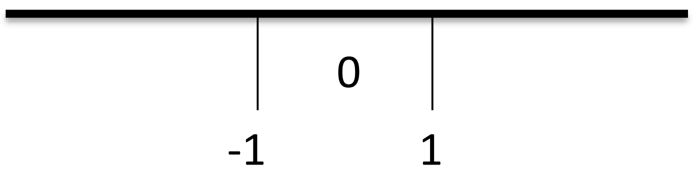
0 = 00000000

What about 10000000?

Major con: Two ways to represent zero.

Two's Complement (signed)

- Borrow nice property from number line:



Only one instance of zero!
Implies: -1 and 1 on either side of it.

Two's Complement

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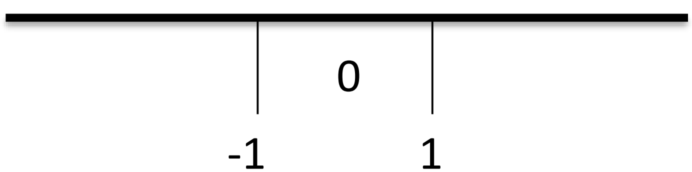

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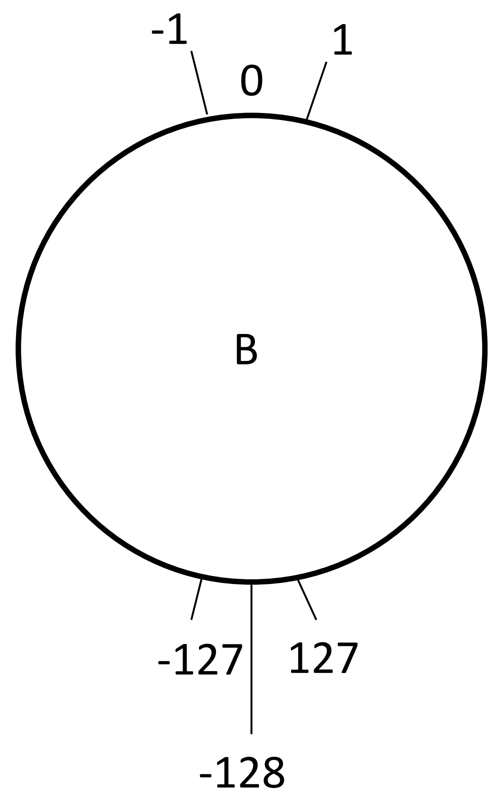


Two's Complement

- Only one value for zero
- With N bits, can represent the range:
  - -2^{N-1} to $2^{N-1} - 1$
- First bit still designates positive (0) /negative (1)

- Negating a value is slightly more complicated:

1 = 00000001, -1 = 11111111

From now on, unless we explicitly say otherwise, we'll assume all integers are stored using two's complement! This is the standard!

Two's Compliment

- Each two's compliment number is now:

$$[-2^{n-1}*d_{n-1}] + [2^{n-2}*d_{n-2}] + \ldots + [2^{1}*d_{1}] + [2^{0}*d_{0}]$$

Note the negative sign on just the first digit. This is why first digit tells us negative vs. positive.

If we interpret 11001 as a two's complement number, what is the value in decimal?

- Each two's compliment number is now:

$[-2^{n-1} * d_{n-1}] \quad + \quad [2^{n-2} * d_{n-2}] \quad +\ldots+ \quad [2^{1} * d_{1}] \quad + \quad [2^{0} * d_{0}]$

A. -2

B. -7

C. -9

D. -25

“If we interpret…”

- What is the decimal value of 1100?

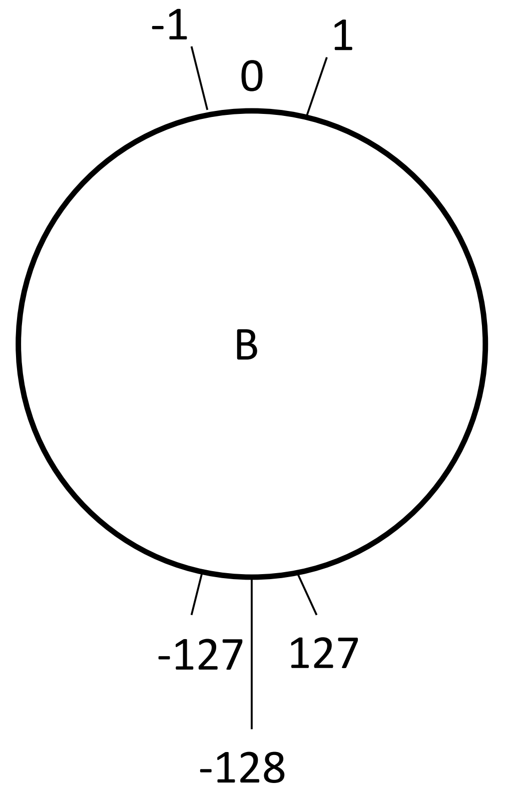
- …as unsigned, 4-bit value: 12 (%u)
- …as signed (two’s comp), 4-bit value: -4 (%d)

- …as an 8-bit value: 12
  (i.e., **0000**1100)

Two's Complement Negation

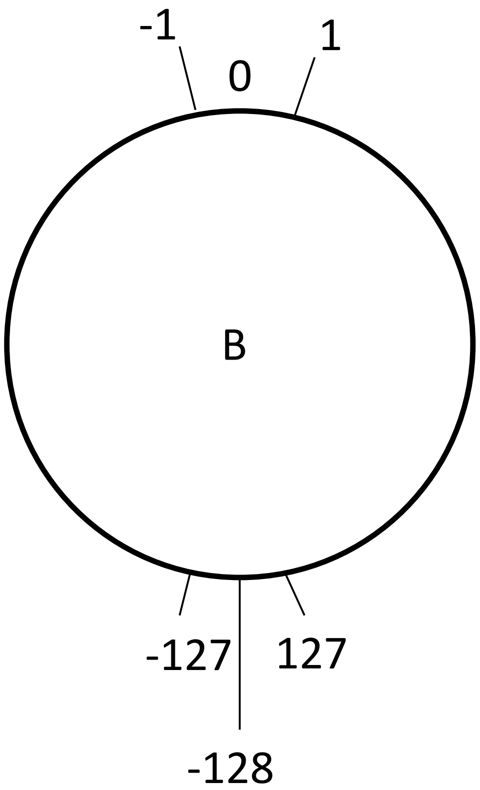
- To negate a value x, we want to find y such that x + y = 0.

- For N bits, $y = 2^N - x$



Negation Example (8 bits)

- For N bits, $y = 2^N - x$
- Negate 00000010 (2)
  - $2^8 - 2 = 256 - 2 = 254$
- Our wheel only goes to 127!
  - Put -2 where 254 would be if wheel was unsigned.
  - 254 in binary is 11111110



Given 11111110, it's 254 if interpreted as <u>unsigned</u> and -2 interpreted as <u>signed</u>.

Negation Shortcut

- A much easier, faster way to negate:
  - Flip the bits (0’s become 1’s, 1’s become 0’s)
  - Add 1

- Negate 00101110 (46)
  - 2^8 - 46 = 256 - 46 = 210
  - 210 in binary is 11010010

Addition & Subtraction

- Addition is the same as for unsigned
  - One exception: different rules for overflow
  - Can use the same hardware for both

- Subtraction is the same operation as addition
  - Just need to negate the second operand…

- 6 - 7 = 6 + (-7) = 6 + (~7 + 1)
  - ~7 is shorthand for “flip the bits of 7”

Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1

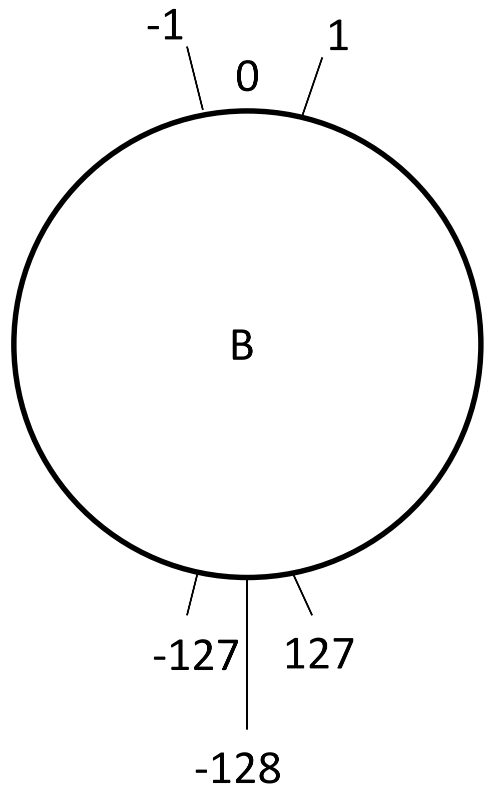
```
input 1 --------------------------------> +-------------+
input 2 --> [possible bit flipper] -->    | ADD CIRCUIT | ---> result
          possible +1 input-------->      +-------------+
```

By switching to two's complement, have we solved this value "rolling over" (overflow) problem?

A. Yes, it's gone.

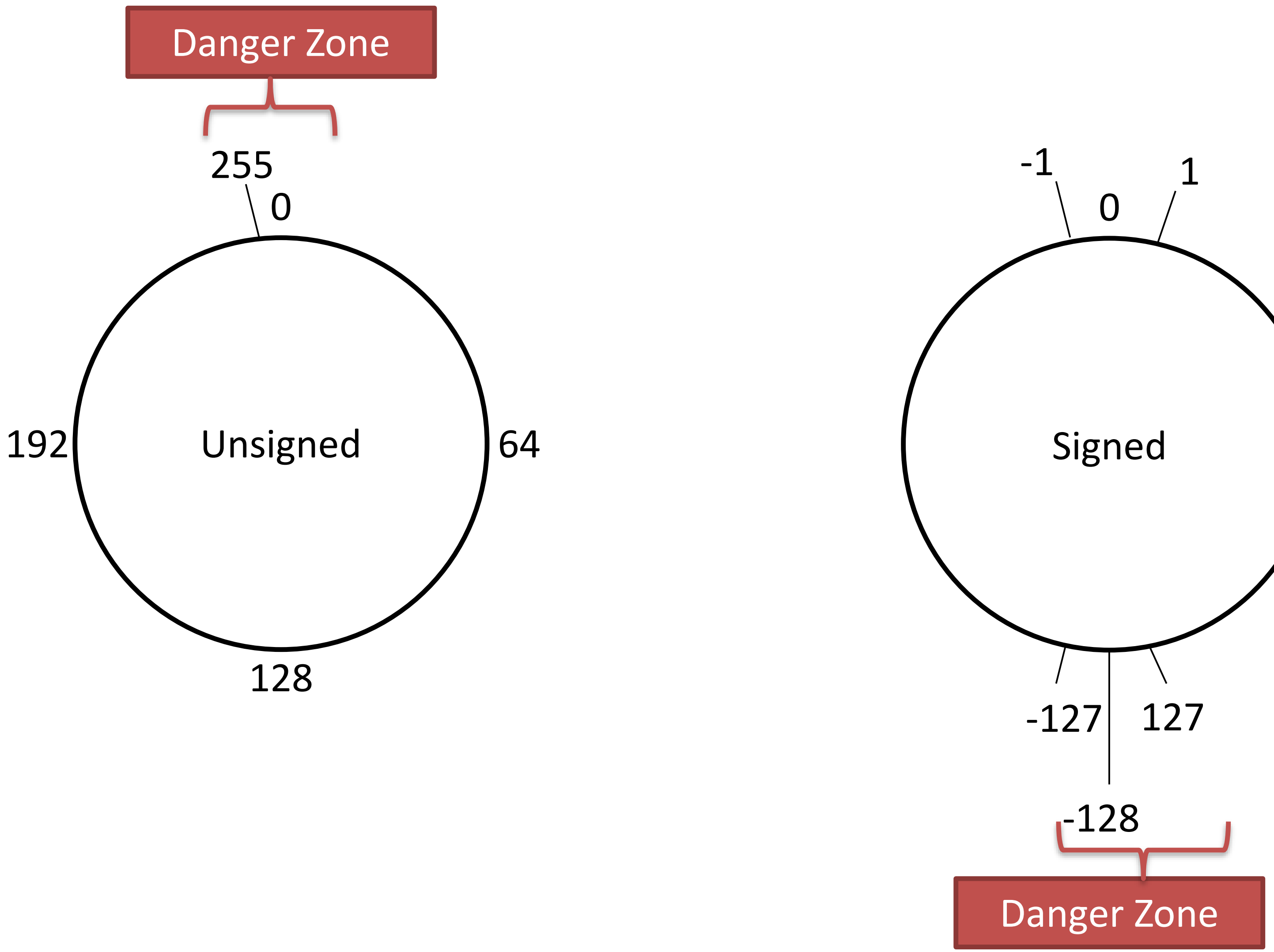
B. Nope, it's still there.

C. It's even worse now.



This is an issue we need to be aware of when adding and subtracting!

Overflow, Revisited

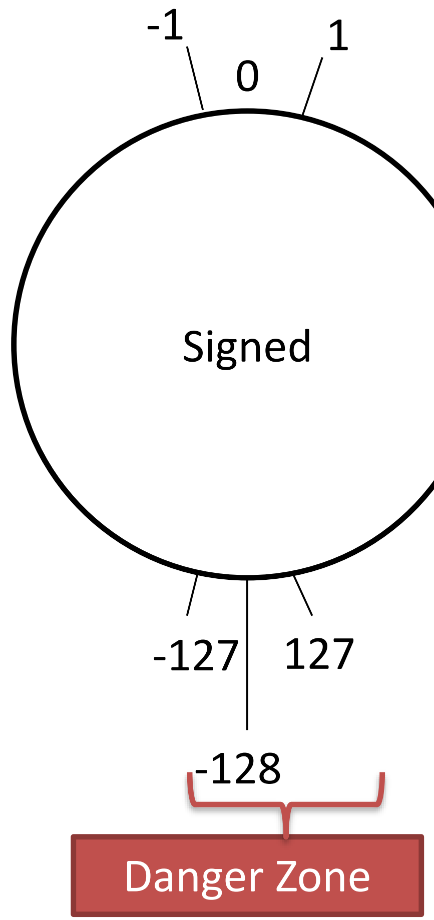


If we add a positive number and a negative number, will we have overflow? (Assume they are the same # of bits)

A. Always

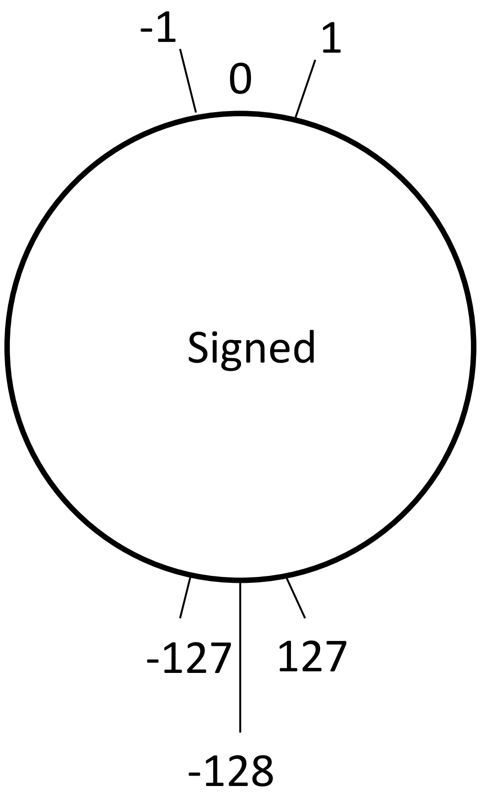
B. Sometimes

C. Never



Signed Overflow

- Overflow: IFF the sign bits of operands are the same, but the sign bit of result is different.
  - Not enough bits to store result!



<u>Signed addition (and subtraction)</u>:

2+-1=1	2+-2=0	2+-4=-2
0010	**0**010	**0**010
+**1**111	+**1**110	+**1**100
1 0001	1 0000	1110

No chance of overflow here - signs of operands are different!

Signed Overflow

- Overflow: IFF the sign bits of operands are the same, but the sign bit of result is different.
    - Not enough bits to store result!

<u>Signed addition (and subtraction)</u>:

2+-1=1	2+-2=0	2+-4=-2	2+7=-7	-2+-7=7
0010	0010	0010	**0**010	**1**110
+1111	+1110	+1100	+**0**111	+**1**001
1 0001	1 0000	1110	**1**001	1 **0**111

Overflow here! Operand signs are the same, and they don't match output sign!

Overflow Rules

- Signed:
  - The sign bits of operands are the same, but the sign bit of result is different.
- Can we formalize unsigned overflow?
  - Need to include subtraction too, skipped it before.

Recall Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1

input 1 -------------------------------->

input 2 --> possible bit flipper --> ADD CIRCUIT ---> result

possible +1 input-------->

Let’s call this +1 input: “Carry in”

How many of these unsigned operations have overflowed?

4 bit unsigned values (range 0 to 15):

						carry-in		carry-out	
Addition (carry-in = 0)									
9 + 11	=	1001	+	1011	+	0	=	1	0100
9 + 6	=	1001	+	0110	+	0	=	0	1111
3 + 6	=	0011	+	0110	+	0	=	0	1001
Subtraction (carry-in = 1)									
6 - 3	=	0110	+	1100 (-3)	+	1	=	1	0011
3 - 6	=	0011	+	1010 (-6)	+	1	=	0	1101

A. 1
B. 2
C. 3
D. 4
E. 5

How many of these <u>unsigned</u> operations have overflowed?

Interpret these as 4-bit unsigned values (range 0 to 15):

Addition (carry-in = 0)

```
                              carry-in  carry-out
                                 ↓        ↓
9 + 11  =   1001 + 1011 + 0 =    1    0100 =  4
9 +  6  =   1001 + 0110 + 0 =    0    1111 = 15
3 +  6  =   0011 + 0110 + 0 =    0    1001 =  9
```

Subtraction (carry-in = 1)

```
                   (-3)
6 -  3  =   0110 + 1100 + 1  =   1    0011 =  3
3 -  6  =   0011 + 1010 + 1  =   0    1101 = 13
                   (-6)
```

A. 1
B. 2
C. 3
D. 4
E. 5

Pattern?

Overflow Rule Summary

- Signed overflow:
  - The sign bits of operands are the same, but the sign bit of result is different.

- Unsigned: overflow
  - The carry-in bit is different from the carry-out.

C_{in}	C_{out}	C_{in} XOR C_{out}
0	0	0
0	**1**	**1**
1	**0**	**1**
1	1	0

So far, all arithmetic on values that were the same size. What if they're different?

Suppose I have an 8-bit value, 00010110 (22), and I want to add it to a signed four-bit value, 1011 (-5). How should we represent the four-bit value?

A. 1101 (don’t change it)
B. 00001101 (pad the beginning with 0’s)
C. 11111011 (pad the beginning with 1’s)
D. Represent it some other way.

Sign Extension

- When combining signed values of different sizes, expand the smaller to equivalent larger size:

```
char y=2, x=-13;
short z = 10;
```

```
z = z + y;

0000000000001010
+       00000010
0000000000000010
```

```
z = z + x;

0000000000000101
+       11110011
1111111111110011
```

Fill in high-order bits with sign-bit value to get same numeric value in larger number of bytes.

Let’s verify that this works

4-bit signed value, sign extend to 8-bits, is it the same value?

0111 ---> 0000 0111 obviously still 7

1010 ----> 1111 1010 is this still -6?

-128 + 64 + 32 + 16 + 8 + 0 + 2 + 0 = -6 yes!

Operations on Bits

- For these, doesn't matter how the bits are interpreted (signed vs. unsigned)
- Bit-wise operators (AND, OR, NOT, XOR)
- Bit shifting

Bit-wise Operators

- bit operands, bit result (interpret as you please)

& (AND) | (OR) ~(NOT) ^(XOR)

```
A     B        A & B     A | B     ~A     A ^ B
-------------------------------------------------
0     0          0         0        1       0
0     1          0         1        1       1
1     0          0         1        0       1
1     1          1         1        0       0
```

```
  01010101        01101010        10101010     ~10101111
| 00100001      & 10111011      ^ 01101001     ---------
----------      ----------      ----------      01010000
  01110101        00101010        11000011
```

More Operations on Bits

- Bit-shift operators: << left shift, >> right shift

```
01010101 << 2  is 01010100
                  2 high-order bits shifted out
                  2 low-order bits filled with 0
01101010 << 4  is 10100000
01010101 >> 2  is 00010101
01101010 >> 4  is 00000110

10101100 >> 2  is 00101011 (logical shift)
               or 11101011 (arithmetic shift)
```

Arithmetic right shift: fills high-order bits w/sign bit

C automatically decides which to use based on type:

signed: arithmetic, unsigned: logical

Up Next

- C programming