Dovetailing and diagonalization

Def: Let \( S \) be a set.
- If there is a bijection \( f: \mathbb{N} \to S \) then \( S \) is countably infinite.
- If \( S \) is finite or countably infinite, then \( S \) is countable.
- If \( S \) is not countable, it is uncountable.

Fact: The union of two countably infinite sets is countably infinite.

Proof (directly):
Let \( A = \{ a_1, a_2, a_3, \ldots \} \) be two countably infinite sets.
\( B = \{ b_1, b_2, b_3, \ldots \} \)

So we know there are two bijective functions \( f_A: \mathbb{N} \to A \)
\( f_B: \mathbb{N} \to B \)

\[ A \cup B = \{ a_1, b_1, a_2, b_2, a_3, b_3, \ldots \} \]

So let's build \( f: \mathbb{N} \to A \cup B \):
\[
\begin{align*}
1 & \to a_1 \\
2 & \to b_1 \\
3 & \to a_2 \\
4 & \to b_2 \\
\vdots & \\
\end{align*}
\]

This is not a set, it might have duplicates!

Proposed fix: remove the duplicates.

\( B \setminus A = \) everything in \( B \) that's not in \( A = \{ c_1, c_2, c_3, \ldots \} \)

Now we can build \( f: \mathbb{N} \to B \setminus A \):
\[
\begin{align*}
1 & \to a_1 \\
2 & \to c_1 \\
3 & \to a_2 \\
4 & \to c_2 \\
\vdots & \\
\end{align*}
\]

onto? Yes, same reason as before.

one-to-one? Yes, we removed too much.
What about the union of 3 countably infinite sets? Still countably infinite.

Fact: The union of countably infinitely many countably infinite sets is countably infinite.

Proof idea (by picture)

Let $S_1, S_2, S_3, \ldots$ be countably many countably infinite sets.

Build an bijection to follow this line but skip duplicates we’ve already covered, it should be odd & one-to-one.

This technique is called **DOVETAILING**.

The technique we saw Monday for \(|\mathbb{N}| \neq |\mathbb{R}|\) is called **DIAGONALIZATION**.

<table>
<thead>
<tr>
<th>digits</th>
<th>$x_1$</th>
<th>500</th>
<th>(10,000)</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>(\pi)</td>
<td>3.</td>
<td>(14159)</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
$x_1 = \pi \approx 3.14159 \ldots$

$\frac{\pi}{2} = 2 \cdot \frac{1}{2} \times \pi = 0.125 \times 0.125 \ldots$

Follow the diagonal to build a number not in this list.