

# CS46 practice problems 8

These practice problems are an opportunity for discussion and trying many different solutions. They are **not counted towards your grade**, and **you do not have to submit your solutions**. The purpose of these problems is to get more comfortable with reasoning and writing about Turing machines, decidability, and recognizability.

1. **Useless variables.** Given a grammar  $G$ , we say that a variable  $V \in G$  is “useless” if there is no string  $w$  for which a possible derivation of  $w$  contains the variable  $V$ . Formulate the problem of finding grammars containing useless variables as a language and show that this language is decidable.
2. **Infinite languages.** Last week we saw that the following language was decidable:

$$INFINITE_{DFA} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) \text{ is an infinite language}\}$$

(See solved problem 4.10 in the book for a clever way of making this argument.)

- (a) Show that  $INFINITE_{CFG}$  is decidable<sup>1</sup>, where:

$$INFINITE_{CFG} = \{\langle G \rangle \mid G \text{ is a context-free grammar and } L(G) \text{ is an infinite language}\}$$

- (b) Show that  $INFINITE_{TM}$  is not decidable, where:

$$INFINITE_{TM} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is an infinite language}\}$$

3. **Classifying languages.** For each of the following languages, is the language decidable?<sup>2</sup> Turing-recognizable? co-Turing-recognizable?

Provided an argument for your answers. (Give the deciders/recognizers that you claim exist, and show why they work; if they do not exist, then prove why not.)

- (a)  $E_{TM} = \{\langle M \rangle \mid L(M) = \emptyset\}$
- (b)  $HUNDRED_{TM} = \{\langle M, w \rangle \mid M \text{ never moves its head past the } 100^{\text{th}} \text{ tape square during its computation on } w\}$

4. **Equal language checking for grammars.**

- (a) Show that  $EQ_{CFG}$  is undecidable.<sup>3</sup>  
Observe that this means an “Automata Tutor for grammars” isn’t coming anytime soon.
- (b) Show that  $EQ_{CFG}$  is co-Turing-recognizable.<sup>4</sup>

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<sup>1</sup>Hint 1: The clever solution for  $INFINITE_{DFA}$  was linked to the pumping lemma for DFAs. How can you use the pumping lemma for context-free languages in a similar way?

Hint 2: The intersection of a context-free language with a regular language is context-free.

<sup>2</sup>Hint: We already know that  $E_{TM}$  it is not decidable.

<sup>3</sup>Hint: Theorem 5.13 shows  $ALL_{CFG}$  is undecidable; you can use this result without proof. (Sipser 5.1)

<sup>4</sup>Hint: Use nondeterminism. (Sipser 5.2)