CS46 practice problems 5

These practice problems are an opportunity for discussion and trying many different solutions. They are **not counted towards your grade**, and **you do not have to submit your solutions.** The purpose of these problems is to get more comfortable with CFGs, PDAs, and the pumping lemma for context-free languages.

1. Use the pumping lemma for context-free languages to show that the following language over $\Sigma = \{a, \#\}$ is not context-free:

$$\{a^n \# a^{2n} \# a^{3n} \mid n \geq 0\}$$

2. Let $L_{\text{eq-len}} = \{x\#y \mid |x| = |y|\}$, where $x, y \in \{a, b\}^*$.
   (a) Design a grammar for $L_{\text{eq-len}}$.
   (b) Describe a PDA for $L_{\text{eq-len}}$.

3. Let $L_{\text{neq-len}} = \{x\#y \mid |x| \neq |y|\}$, where $x, y \in \{a, b\}^*$.
   (a) Design a grammar for $L_{\text{neq-len}}$.
   (b) Describe a PDA for $L_{\text{neq-len}}$.

4. Let $L_{\text{eq}} = \{x\#y \mid x = y\}$, where $x, y \in \{a, b\}^*$.

   Show that L_{eq} is not context-free using the pumping lemma for context-free languages.

5. Let $L_{\text{neq}} = \{x\#y \mid x \neq y\}$, where $x, y \in \{a, b\}^*$.
   (a) First, design a grammar or PDA for the language $\{a^p b^q b^p a^q \mid p, q \geq 0\}$. (Hint: it should be pretty simple.)
   (b) (Hint step. Next, figure out how to break up L_{neq} into subsets. Some subsets you should already have solved (above). For the remaining subset, find a way to write it which looks a lot like part (a).)
   (c) Design a grammar for L_{neq}.
   (d) Describe a PDA for L_{neq}.

6. Use the pumping lemma for context-free languages to show that

$$\{w\#t \mid w \text{ is a substring of } t, \text{ and } w, t \in \{a, b\}^*\}$$

   is not context-free.

7. If A and B are languages, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$.
   (a) Show that if A and B are regular languages, then $A \diamond B$ is a context-free language. You will probably want to describe a PDA instead of a grammar, and use nondeterminism. You do not need to give a completely formal definition of your PDA, but instead describe at a high level how it works.

(b) If A and B are regular languages, is $A \diamond B$ regular? If so, give a proof. If not, give an example of two regular languages A and B for which $A \diamond B$ is not regular, and prove your claim.

(c) **(challenge)** If A is regular and B is context-free, what can you say about $A \diamond B$? Is it regular? Is it context-free?

Extra practice problems!

We now have a reasonable collection of interesting problems. For each of these languages that we've encountered recently, is it

(i) regular?

(ii) context-free, but not regular?

(iii) not even context-free?

Support your choice with a construction or proof. (You may have already done parts of this problem in the course of last week's lab/homework.) Remember that we have many labor-saving theorems and already-proven facts that you can cite. If you've already done a proof using a pumping lemma, try to do it again with closure properties.

8. $L_1 = \{w\overline{w} \mid \overline{w}$ is w with all as flipped to bs and all bs flipped to as$\}$ where $\Sigma = \{a, b\}$.
9. $L_2 = \{a^k u a^k \mid k \geq 1$ and $u \in \Sigma^*\}$ where $\Sigma = \{a, b\}$.
10. $L_3 = \{a^k b u a^k \mid k \geq 1$ and $u \in \Sigma^*\}$ where $\Sigma = \{a, b\}$.
11. $L_4 = \{w \mid w$ is unary for 10^n for some $n \geq 0\}$ where $\Sigma = \{1\}$.
12. $L_5 = \{w \mid w$ is decimal for 10^n for some $n \geq 0\}$ where $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
13. $L_6 = \{a^n b^m a^m b^n \mid m, n \geq 0\}$ where $\Sigma = \{a, b\}$.
14. $L_7 = \{a^{m-n} \mid \frac{m}{n} = 5\}$ where $\Sigma = \{a, b\}$.
15. $L_8 = \{a^m b^n \mid m$ and n are prime factors of some integer $\leq 2020\}$ where $\Sigma = \{a, b\}$.
16. $L_9 = \{w \mid w$ is not a palindrome$\}$ where $\Sigma = \{a, b\}$.
17. $L_{10} = \{w \mid w = x_1\#x_2\# \cdots \#x_k$ for $k \geq 0$, each $x_i \in L(a^*)$, and $x_i \neq x_j$ for $i \neq j\}$, where $\Sigma = \{a, \#\}$.