W11L1: P and big O notation

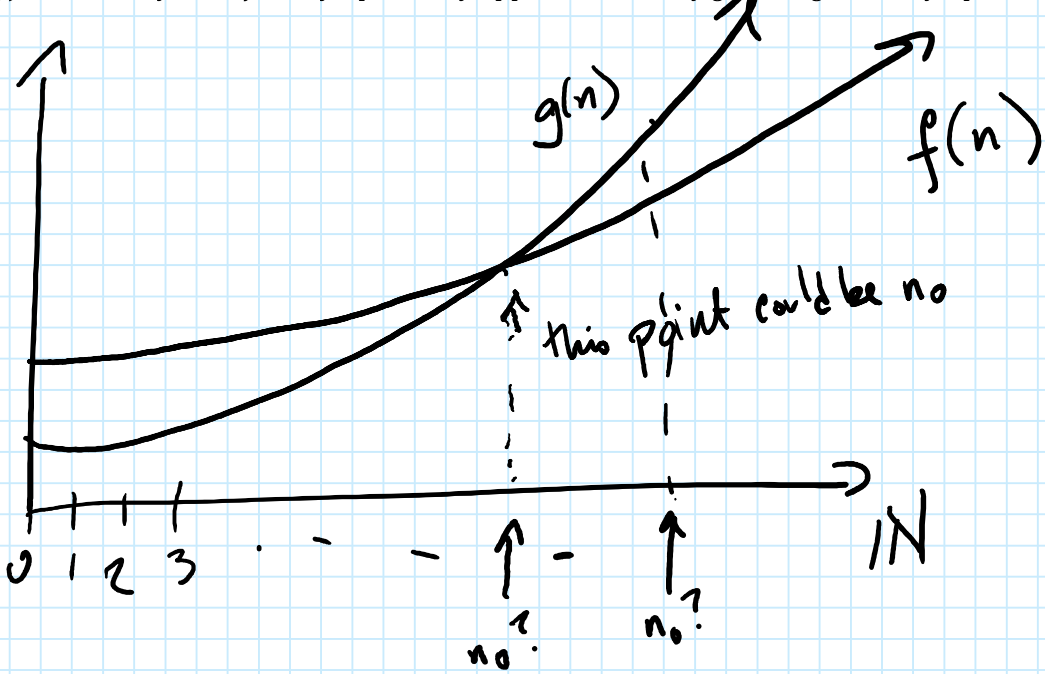
Monday, April 13, 2020 9:21

Announcements:

- quiz this week on gradescope
- no homework this week
- midterm 2 next week "in lab" (on gradescope)

We now want to discuss runtime: how long does it take a Turing machine to reach an answer?

Definition:
Let $f, g: \mathbb{N} \to \mathbb{R}^+$ be two functions.
We say that $f(n)$ is $O(g(n))$ if there exists a constant $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$.
If f is $O(g)$ then we say that "f is asymptotically upper-bounded by g" and "g is an asymptotic upper bound for f."



Problem 1: Prove that $f(n) = 2n^2 + 7n + 16$ is $O(g(n))$ where $g(n) = n^2$.

Pick $c = 25$ and $n_0 = 1$.
Want to show that $2n^2 + 7\,n + 16 \leq 25 \cdot (n^2)$ for $n \geq n_0 = 1$.

$c \cdot g(n) = 2n^2 + 7n^2 + 16n^2 \geq 2n^2 + 7n + 16n^2$ b/c $n \geq 1$

$\geq 2n^2 + 7n + 16$ b/c $n \geq 1$

$= f(n)$

$\Rightarrow f(n)$ is $O(g(n))$. ∎

Problem 2: Order these 3 functions by increasing $O(\cdot)$ asymptotic growth:

- $f(n) = 100n$
- $g(n) = \frac{n^{10}}{}$

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- $f(n) = 100n$
- $g(n) = \frac{n^{10}}{6}$
- $h(n) = 4n^2$

Claim: f is $O(h)$ and h is $O(g)$.

Heuristic: look at the highest-order term

Theorem: If f, g, and h are all functions and $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$.
idea: $O(\cdot)$ is transitive

We are going to use $O(\cdot)$ to compare the efficiency of different Turing machines to solve the same problem.

The "speed" of a Turing machine is the number of steps used in computation (up until it halts). This runtime is a function of input size: $f(n)\colon \mathbb{N} \to \mathbb{N}$ will be the number of steps a Turing machine takes (until it halts) when given an input of $length = n$.

If we have a Turing machine M which decides $L \subseteq \Sigma^*$ then we can define a function
$T_M(w) =$ the number of steps M takes on input w

We are interested in the runtime of M, which is the **worst-case** runtime, defined as:
$T_M(n) =$ maximum number of steps M takes on any input w where $|w| = n$.