# Notes on the Myhill-Nerode Theorem

The purpose of this note is to give some details of the Myhill-Nerode Theorem and its proof, neither of which appear in the textbook. This theorem will be a useful tool in designing DFAs, as well as in characterizing the regular languages.

**Definition 1.** *Let $L \subseteq \Sigma^*$ be any language, and $x, y \in \Sigma^*$ be any strings. We say "$x$ is* **equivalent** *to $y$ with respect to $L$", written $x \approx_L y$ iff, for any $z \in \Sigma^*$,*

$$xz \in L \iff yz \in L$$

Observe: $\approx_L$ is an equivalence relation: it is reflexive, symmetric, and transitive. You should check this to convince yourself.

We will be interested in decomposing a language into its equivalence classes, which we write as:

$$[x] = \{y \mid x \approx_L y\} \text{ is the equivalence class of } x \text{ under } \approx_L$$

For example, consider $L = \{w \mid w \text{ is of even length}\} \subseteq \{a\}^*$. The equivalence classes of $L$ are:

- $[\epsilon] = \{\epsilon, aa, aaaa, aaaaaa, \ldots\} = [aa] = [aaaa] = L$
- $[a] = \{a, aaa, aaaaa, \ldots\} = \{w \mid w \text{ is of odd length}\} = [aaa] = [aaaaaaa] = \cdots$

Notice that $[a] = \bar{L}$ is the complement of $L$, so this is all the equivalence classes. Also, note that $[\epsilon] = [aa] = [aaaa]$ (and so on), so we can call an equivalence class by many different names.

**Theorem 2** (Myhill-Nerode Theorem)**.** *$L$ is regular if and only if $\approx_L$ has finitely many equivalence classes.*

The idea is that each equivalence class will correspond to a state of the DFA. (This makes sense, since if $x$ and $y$ are in the same equivalence class, then for any string $z$ we concatenate to the end, $xz \in L \iff yz \in L$ — that is, we want the DFA to either accept both $xz$ and $yz$ or reject both of them. This will correspond to starting from the same state, and then processing the characters of string $z$.)

*Proof.* There are two directions of the "if and only if".

$\Leftarrow$: If $L$ is regular, then there is a DFA recognizing $L$ which has finitely many states. Each state represents an equivalence class (of strings that reach that state). Consider two strings $x$ and $y$ which both finish in some state $q_i$. Then for any string $z$, the computation on $xz$ will end up in the same state as the computation for $yz$, namely, whatever state the DFA reaches when it starts in state $q_i$ and sees string $z$.

Since there are finitely many states and each state represents an equivalence class, there are finitely many equivalence classes.

$\Rightarrow$: If $L$ has finitely many equivalence classes, then there is a DFA recognizing $L$ with exactly that many states. We can construct it as follows. Define DFA $M = (Q, \Sigma, \delta, q_0, F)$:

$$\begin{aligned} Q &= \{[x] \mid x \in \Sigma^*\} \\ q_0 &= [\epsilon] \\ F &= \{[x] \mid x \in L\} \\ \delta([x], \sigma) &= [x\sigma] \text{ for } [x] \in Q, \sigma \in \Sigma \end{aligned}$$

Note: $\delta$ is well-defined because $x \approx_L y$ iff $x\sigma \approx_L y\sigma$.

Some observations to make:

- for any string $x$, it is in some equivalence class $[x]$ and it will end up in the state corresponding to $[x]$
- for any string $x$, if $x \in L$ then the state corresponding to $[x]$ is a final state (by the construction rule given above), so $x$ will be accepted
- for any string $x \notin L$, the state corresponding to $[x]$ is not a final state. Why?
  A tiny proof-by-contradiction:
  Suppose $x \notin L$ but the state $q$ corresponding to $[x]$ was in $F$.
  Because $q \in F$, it must be that $[x] = [y]$ for some $y \in L$ (by the construction rule given above for set $F$).
  If these two equivalence classes are equal, that means $x \approx_L y$ (by definition of equivalence classes).
  Thus for all $z$, $xz \in L \iff yz \in L$.
  Take $z = \epsilon$. Then we have $x\epsilon = x \in L$ is FALSE but $y\epsilon = y \in L$ is TRUE. Contradiction! $\Rightarrow\Leftarrow$

Thus the DFA given by this construction recognizes the language $L$. $\square$

**Corollary 3.** *Let $L$ be a language with $k \in N$ equivalence classes under $\approx_L$. Then every DFA recognizing $L$ has at least $k$ states.*

And note, for $L$ with $k$ equivalence classes, the above construction gives a DFA with exactly $k$ states — a minimal DFA, the smallest one possible.[1]

**Practice problem 1:** Consider again the example language: $L = \{w \mid w \text{ is of even length}\} \subseteq \{a\}^*$. The equivalence classes of $L$ are:

- $[\epsilon] = \{\epsilon, aa, aaaa, aaaaaa, \ldots\} = [aa] = [aaaa] = L$
- $[a] = \{a, aaa, aaaaa, \ldots\} = \{w \mid w \text{ is of odd length}\} = [aaa] = [aaaaaaa] = \cdots$

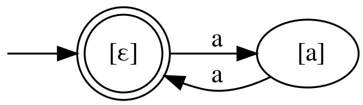
The Myhill-Nerode Theorem says that because $L$ has finitely many equivalence classes[2], it should be a regular language. Can you use the insight of the proof to come up with a (very, very simple) DFA that accepts this language $L$? (Ideally, you would only have as many states as there are equivalence classes.) Answer on the next page.

**Practice problem 2:** Consider the language:

$$L = \{w \in \{0,1\} \mid w \text{ represents a number divisible by 3 in binary notation}\}$$

How many equivalence classes does this $L$ have? What are they? Can you come up with a DFA to recognize this language?

[1] Fun thought experiment and proof-writing practice: why would any smaller DFA *not* be able to recognize $L$?
[2] Check for yourself: how many are there? 2. Sanity check: is 2 finite? Yeah.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