CS46 practice problems 10

These practice problems are an opportunity for discussion and trying many different solutions. It is **not counted towards your grade**, and **you do not have to submit your solutions**. The purpose of these problems is to get more comfortable with reasoning and writing about P, NP, and polynomial-time reductions.

If you are stumped or looking for guidance, **ask**.

1. Recall that a **vertex cover** in a graph $G$ is a subset of vertices where every edge of $G$ has at least one endpoint in the subset.

   $$\text{VERTEXCOVER} = \{\langle G, k \rangle \mid G \text{ has a } k\text{-node vertex cover} \}$$

   An **independent set** in a graph $G$ is a subset of vertices with no edges between them.

   $$\text{INDEPENDENTSET} = \{\langle G, k \rangle \mid G \text{ contains an independent set of } k \text{ vertices} \}$$

   Show that $\text{INDEPENDENTSET} \leq_p \text{VERTEXCOVER}.$

2. A Boolean formula is in **conjunctive normal form** (CNF) if it is written as the conjunction of clauses, for example:

   $$(x_1 \lor x_2) \land (\overline{x}_2 \lor \overline{x}_3 \lor x_4) \land (x_5 \lor \overline{x}_1 \lor x_6) \land (x_3)$$

   (Recall that a **literal** is a Boolean variable or a negated Boolean variable, like $x$ or $\overline{x}$, and a **clause** is a disjunction of literals, like $x \lor y \lor \overline{z}$. The symbol “$\lor$” means “or”; the symbol “$\land$” means “and”.) A formula is **satisfiable** if there is a truth assignment (giving a truth value to each variable) which makes the entire formula evaluate to **TRUE**.

   (a) (Sipser 7.24a) Define the language:

   $$L = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable CNF formula where each variable appears at most twice} \}$$

   Show that $L \in P$.

   (b) (Sipser 7.43) For a CNF formula $\phi$ with $m$ variables and $c$ clauses, show you can construct in polynomial time an NFA with $O(cm)$ states that accepts all nonsatisfying assignments, represented as binary strings of length $m$.

   (This implies that if $P \neq NP$, then NFAs cannot be minimized in polynomial time.)

3. (Sipser 7.29) A **coloring** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color.

   Define **3-COLOR** as:

   $$3-COLOR = \{\langle G \rangle \mid G \text{ is colorable with three colors} \}$$

   Show that **3-COLOR** is NP-complete. (Hint: reduce from 3-SAT and use the subgraphs given in the textbook hint, page 325.)