Reducing $\text{HAMPATH} \leq_p \text{TRAVELLING SALESMAN PROBLEM}$

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April 12, 2017
**Hampath**

\[ \text{Hampath} = \left\{ \langle G, s, t \rangle \mid G \text{ is a graph with a path from vertex } s \text{ to vertex } t \text{ which visits each other vertex exactly once} \right\} \]

**TravellingSalesmanProblem**, abbreviated TSP

\[ \text{TSP} = \left\{ \langle C, d, b \rangle \mid C \text{ is a list of cities} \right. \\
\text{ } \text{ } \text{ } \text{ } \text{ } \left. d : C \times C \to \mathbb{N} \text{ gives the distance between cities} \right\}
\text{There is a round-trip path which visits each non-home city exactly once, and has total distance } \leq b. \]

Wanted to show that \text{Hampath} \leq_p \text{TSP}.
The reduction \( f \)

On input \( \langle G, s, t \rangle \):

1. If \( \langle G, s, t \rangle \) is badly formatted, return \( \varepsilon \) (a badly-formatted string for TSP).
2. Create \( C \), which includes one city \( c_v \) for each vertex \( v \) in \( G \), plus one more home city \( c_{\text{home}} \).
3. Count the vertices in \( G \). Call this number \( n \). Set \( b = n + 1 \).
4. Define \( d(c_v, c_w) \) as follows:
   - \( = 1 \), if \( v \) and \( w \) are vertices in \( G \) and have an edge between them.
   - \( = b + 1 \), if \( v \) and \( w \) are vertices in \( G \) and don’t have an edge between them.
   - \( = 1 \), if one city is \( c_{\text{home}} \) and the other is \( c_s \) or \( c_t \).
   - \( = b + 1 \), if one city is \( c_{\text{home}} \) and the other is anything else.
5. Return \( \langle C, d, b \rangle \).
The reduction \( f \) is polynomial-time computable.

On input \( \langle G, s, t \rangle \):

1. If \( \langle G, s, t \rangle \) is badly formatted, return \( \varepsilon \) (a badly-formatted string for TSP).
   \[ O(\|\langle G, s, t \rangle\|) \]

2. Create \( C \), which includes one city \( c_v \) for each vertex \( v \) in \( G \), plus one more home city \( c_{\text{home}} \).
   \[ O(\text{number of vertices}) \leq O(\|\langle G, s, t \rangle\|) \]

3. Count the vertices in \( G \). Call this number \( n \). Set \( b = n + 1 \).
   \[ O(\|\langle G, s, t \rangle\|) \]

4. Define \( d(c_v, c_w) \) as follows:
   - \( = 1 \), if \( v \) and \( w \) are vertices in \( G \) and have an edge between them.
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   - \( = 1 \), if one city is \( c_{\text{home}} \) and the other is \( c_s \) or \( c_t \).
   - \( = b + 1 \), if one city is \( c_{\text{home}} \) and the other is anything else.
   \[ O((\text{number of vertices in } G+1)^2) \leq O(\|\langle G, s, t \rangle\|^2) \]

5. Return \( \langle C, d, b \rangle \).
   \[ O(\|\langle G, s, t \rangle\| + \|\langle G, s, t \rangle\|^2) \]

Total: definitely polynomial in \( \|\langle G, s, t, \rangle\| \).
If $w = \langle G, s, t \rangle \in \text{HAMPATH}$, then $f(w) \in \text{TSP}$.

Call $f(w) = \langle C, d, b \rangle$.

If $w \in \text{HAMPATH}$, then it is properly encoded, so it passes the check on line 1, does the rest of the steps, and returns some $\langle C, d, b \rangle$.

Since $\langle G, s, t \rangle \in \text{HAMPATH}$, there must be some path from $s$ to $t$ which visits every vertex exactly once:

$s, v_1, v_2, \ldots, v_{n-3}, v_{n-2}, t$

Consider the roundtrip which is:

$C_{\text{home}}, C_s, C_{v_1}, \ldots, C_{v_{n-2}}, C_t, C_{\text{home}}$

This visits each city exactly once, then returns home. Each “hop” is distance 1 by our construction of $d$, so the total distance is $n + 1 = b$.

Thus there exists a roundtrip within the distance bound!
If $f(w) \in \text{TSP}$, then $w = \langle G, s, t, \rangle \in \text{HAMPATH}$. 

If $f(w) \in \text{TSP}$, then it is formatted correctly, so it came from a correctly formatted $w = \langle G, s, t \rangle$.

Since $f(w) \in \text{TSP}$, there is some roundtrip from $c_{\text{home}}$ visiting each other city exactly once:

roundtrip: $c_{\text{home}}, c_1, c_2, \ldots, c_{n-1}, c_n, c_{\text{home}}$

This roundtrip has total distance $\leq b$.

What can $c_1$ and $c_n$ be? If $c_1$ is not $c_s$ or $c_t$, then the distance $d(c_{\text{home}}, c_1) = b + 1$ would mean that our total roundtrip distance is $> b$. Thus $c_1$ must be $c_s$ or $c_t$. (Similarly, $c_n$ must be $c_s$ or $c_t$.)

roundtrip: $c_{\text{home}}, c_s, c_2, \ldots, c_{n-1}, c_t, c_{\text{home}}$
If \( f(w) \in \text{TSP} \), then \( w = \langle G, s, t, \rangle \in \text{HAMPATH} \).

If \( f(w) \in \text{TSP} \), then it is formatted correctly, so it came from a correctly formatted \( w = \langle G, s, t, \rangle \).

Since \( f(w) \in \text{TSP} \), there is some roundtrip from \( c_{\text{home}} \) visiting each other city exactly once, which we can rewrite as:

\[
\text{roundtrip: } c_{\text{home}}, c_s, c_2, \ldots, c_{n-1}, c_t, c_{\text{home}}
\]

This roundtrip has total distance \( \leq b \).

Every “hop” along this roundtrip has distance either 1 or \( b + 1 \), by our construction of \( d \). But for the total distance to be \( \leq b \), they must each be \( = 1 \). By our construction, we know that this means the corresponding vertices in \( G \) have an edge between them.

Here’s a path from \( s \) to \( t \) in \( G \) which visits each vertex exactly once:

\[
s, v_1, v_2, \ldots, v_{n-1}, t
\]

Thus \( \langle G, s, t, \rangle \in \text{HAMPATH} \).
Proving $A \leq_p B$

1. Give a polynomial-time computable function $f$ from strings in $A$ to strings in $B$.
2. Argue that $f$ is polynomial-time computable.
3. Show that if $w \in A$, then $f(w) \in B$.
4. Show that if $f(w) \in B$, then $w \in A$.

Things to think about:

- Is $f(w)$ really computable in time polynomial in $|w|$?
- What if $w \notin A$ because it is formatted incorrectly?
- How to “change” the problem $\in A$ into the problem $\in B$. 

Hampath $\leq_p$ TSP