CS46 Homework 8

This homework is due at 11:59pm on Tuesday, March 28. For this homework, you will work with a partner. It’s ok to discuss approaches at a high level, but most of your discussions should just be with your partner. The only exception to this rule is work you’ve done with another student while in lab. In this case, note who you’ve worked with and what parts were solved during lab.

Your partnership’s write-up should be your own: do not share it, and do not read other people’s write-ups. Please refer to the course webpage or ask me any questions you have about this policy.

0. Before final submission, make sure to fill out the README file.

1. **Classifying languages.** Consider the following languages.
   
   (a) \( ALL_{TM} = \{ \langle M \rangle \mid L(M) = \Sigma^* \} \)
   
   (b) \( ODD_{TM} = \{ \langle M \rangle \mid L(M) \text{ contains no strings of even length} \} \)

   For each, is the language decidable? Turing-recognizable? co-Turing-recognizable? Provided an argument for your answers. (Give the deciders/recognizers that you claim exist, and show why they work; if they do not exist, then prove why not.) You may want to consider these properties in a different order.

2. (Sipser 5.13) A **useless state** in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states.
   
   (a) Formulate this problem as a language.
   
   (b) Show that this language is undecidable.

3. **Homomorphisms again!** Recall the definition of homomorphism: A **homomorphism** is a function \( f : \Sigma \to \Gamma^* \) from one alphabet to strings over another alphabet. We extend \( f \) to operate on strings by defining \( f(w) = f(w_1)f(w_2)\cdots f(w_n) \) where \( w = w_1w_2\cdots w_n \) and each \( w_i \in \Sigma \). We further extend \( f \) to operate on languages by defining \( f(\epsilon) = \epsilon \) and \( f(A) = \{ f(w) \mid w \in A \} \), for any language \( A \).
   
   (a) Show that the decidable languages are not closed under homomorphism. (That is, give an example language \( A \) and homomorphism \( f \) such that \( A \) is decidable, but \( f(A) \) is not decidable.)
   
   (b) A homomorphism is called **nonerasing** if it never maps a character to \( \epsilon \). (Equivalently, \( |f(\sigma)| \geq 1 \) for all \( \sigma \in \Sigma \).) Prove that the decidable languages are closed under nonerasing homomorphisms.

4. Two disjoint languages \( A \) and \( B \) are **decidably separable** if there is a decidable language \( C \) such that \( A \subseteq C \) and \( C \cap B = \emptyset \).
(a) Give two example languages which are decidably separable.

(b) Give two example disjoint languages which are *not* decidably separable.

(c) If $A$ and $B$ are disjoint languages which are both co-Turing-recognizable, show that $A$ is decidably separable from $B$.

(d) **Extra credit.** Given a Turing-recognizable language $A$, let $	ext{machine}(A) = \{ \langle M \rangle \mid L(M) = A \}$. Show that if $A$ and $B$ are Turing-recognizable languages and $A \subsetneq B$, then $	ext{machine}(A)$ is not decidably separable from $	ext{machine}(B)$. 