Max-cut approximation

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Problem: max-cut
input: graph $G = (V, E)$ and nonnegative edge weights w_e for each $e \in E$
output: partition (A, B) of V which maximizes weight of the cut, defined as

$$w(A,B) = \sum_{\substack{e=\{u,v\}\in E \\ u\in A \\ v \in B}} w_e$$

Max-Cut-approx $(G=(V,E), \{w_e\}_{e\in E})$

$A = B = \{\}$
for $v \in V$:
  flip a coin
  if heads, add v to A
  if tails, add v to B
return (A,B)

SANITY CHECKS
halts? yes
runtime? $O(n)$
valid output? yes

Claim: This alg returns a cut with expected approx ratio 2.

Proof: (direct) Let (A,B) be the cut returned by this alg.
Let (A^*, B^*) be the max cut.

$$w(A^*, B^*) = \sum_{\substack{e \text{ edge} \\ \text{across} \\ A^*, B^* \text{ cut}}} w_e \leq \sum_{e \in E} w_e$$

the most edges across (A^*, B^*) is ALL of the edges

Let $X_{uv} = \begin{cases} 1, & \text{if edge } \{u,v\} \in E \text{ crosses the } (A,B) \text{ cut} \\ 0, & \text{otherwise} \end{cases}$

This is an random variable. $\mathcal{X} = \{0, 1\}$

μ the distribution of X_{uv}:

possibilities	probability	
$u\in A$, $v \in A$	1/4	$P_R[X_{uv} = 0] = P_R[\text{edge } \{u,v\} \text{ doesn't cross } A,B \text{ cut}] = 1/2$
$u \in B$, $v \in B$	1/4	
$u \in A$, $v \in B$	1/4	$P_R[\text{edge } \{u,v\} \text{ crosses } A,B \text{ cut}] = 1/2 = P_R[X_{uv} = 1]$
$u \in B$, $v \in A$	1/4	

$$\mathbb{E}[X_{uv}] = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2 \quad (*)$$

KEY setup for making the analysis come together

$$\mathbb{E}[w(A,B)] = \mathbb{E}\Big[\sum_{\substack{e \text{ across} \\ A,B \text{ cut}}} w_e\Big]$$ by def of weight of a cut

$$= \mathbb{E}\Big[\sum_{\substack{e \text{ across} \\ A,B \text{ cut}}} w_e \cdot 1\Big]$$ math

$$= \mathbb{E}\Big[\sum_{\substack{e \text{ across} \\ A,B \text{ cut}}} w_e \cdot X_e\Big]$$ because $X_e = 1$ for all edges e across A,B cut

$$= \mathbb{E}\Big[\sum_{\substack{e \text{ across} \\ A,B \text{ cut}}} w_e \cdot X_e \Big]$$ across A,B cut

$$= \mathbb{E}\Big[\sum_{e \in E} w_e \cdot X_e \Big]$$ because $X_e = 0$ for all edges not across A,B cut, so we just added some zero terms

$$= \sum_{e \in E} \mathbb{E}\big[w_e \cdot X_e \big]$$ by linearity of expectation

$$= \sum_{e \in E} w_e \cdot \underbrace{\mathbb{E}[X_e]}_{= 1/2 \text{ by } (*) \text{ above}}$$ since w_e is constant

$$= \sum_{e \in E} w_e \cdot \frac{1}{2}$$

$$\mathbb{E}\big[w(A,B)\big] = \frac{1}{2} \cdot \sum_{e \in E} w_e$$

$$\underbrace{w(A^*, B^*) \leq \sum_{e \in E} w_e = 2 \cdot \mathbb{E}\big[w(A,B)\big]}$$

because (A^*, B^*) is the max weight cut

$$\frac{1}{2} w(A^*, B^*) \leq \mathbb{E}\big[w(A,B)\big] \leq w(A^*, B^*)$$

So the expected approx ratio is 2. $\blacksquare$

Brainstorm for next class:

Max 3 SAT:

input: n variables $x_1, \dots, x_n$
m clauses of size 3

output: a truth assignment which maximizes # of satisfied clauses