CS41 Homework 6

This homework is due at 11:59PM on Saturday, October 9. Note the unusual due date. This is a 7 point homework. Write your solution using LaTeX. Submit this homework using **github** as a **.tex** file. This is a **partnered homework**. You should primarily be discussing problems with your homework partner.

It's ok to discuss approaches at a high level with others. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you've done with a lab teammate *while in lab*. In this case, note (in your **homework submission poll**) who you've worked with and what parts were solved during lab.

1. **Changing edge costs.** (K & T 4.2) For each of the following two statements, decide whether it is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   (a) Suppose we are given an instance of the Minimum Spanning Tree (MST) problem on a graph $G = (V, E)$, with edge costs that are all positive and distinct. Let T be a minimum spanning tree for this instance. Now, suppose we replace each cost c_e by its square c_e^2, thereby creating a new instance of the problem with the same graph but different costs. TRUE or FALSE? T must still be a minimum spanning tree for this new instance.

   (b) Suppose we are given an instance of the Shortest $s \rightsquigarrow t$ Path Problem on a directed graph $G = (V, E)$, with edge costs that are all positive and distinct. Let P be a a minimum-cost $s \rightsquigarrow t$ path for this instance. Now, suppose we replace each edge cost c_e by its square c_e^2, thereby creating a new instance of the problem with the same graph but different costs.
   TRUE or FALSE? P must still be a minimum $s \rightsquigarrow t$ path for this new instance.

2. **Making change.** Consider the problem of making change for n cents out of the fewest number of coins. Assume that n and the coin values are positive integers (cents).

   (a) Describe a greedy algorithm to solve the problem using the US coin denominations of quarters (25), dimes (10), nickels (5), and pennies (1). Prove your algorithm is optimal.

   (b) Suppose the country of Algorithmland uses denominations that are powers of c for some integer c. This country uses $k+1$ denominations of $c^0, c^1, \ldots, c^k$. Show that your greedy algorithm works in Algorithmland as well.

   (c) Design a currency system of your choosing with at least three coin denominations such that your greedy algorithm does *not* yield a minimum number of coins for some amount of m cents. Assume that one of your denominations has value one (you must have a penny), so a solution exists for all values of m.

3. **(extra challenge)** One reason why the United States has the coin denominations it has is to make it easier to make change using a small number of coins. However, is the US set of coin denominations the optimal set of denominations?

   Perhaps some other set of denominations is better. In this problem, design and analyze an algorithm to determine the optimal set of coin denominations. Given an integer N, say the *change efficiency* of a set of denominations of coins as the average number of coins required to make change, where the average is taken over all $1 \leq m \leq N$.

For example, using the standard US coin denominations $\{1, 5, 10, 25\}$, the coin efficiency given $N = 5$ is 2.2, since it takes m pennies to make change for values $1 \le m \le 4$, and one nickel to make change for value $m = 5$.

Design and analyze an efficient algorithm that takes as input positive integers N and k and returns a set of k denominations that minimizes the coin efficiency for N.