CS41 Lab 12

November 25 2019

This week's lab focuses on solving linear programs. In this exercise, you will walk through the process of optimally solving a LP.

Recall the linear program from lecture this morning:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + 2x_2 + 3x_3 \\ \text{subject to:} \quad & x_1 + x_2 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \end{aligned}$$

Note: all variables should be nonnegative. Throughout this lab writeup, we'll suppress writing these constraints for brevity.

1. First, as a warm-up, solve the following linear program.

$$\begin{aligned} \text{Minimize} \quad & 7 + x_1 + 2x_2 \\ \text{subject to:} \quad & x_3 = 1 + 3x_1 - 4x_2 \\ & x_4 = 3 - 2x_1 + x_2 \\ & x_5 = 2 + x_1 + x_2 \end{aligned}$$

   Hint: The answer should be *reallllllllly* simple.

2. The linear program in part (1) is easy to solve for three reasons: (i) each constraint is an equality, (ii) the constants on the right-hand side of each constraint are positive, and (iii) the variables in the objective function all have positive coefficients. Our goal for the rest of the problem is to transform the LP we'd like to solve so it has all three conditions.

   First, handle condition (i). It's possible to replace any inequality with an equality by adding an additional nonnegative variable called a *slack variable*. For example, if the LP has a constraint $x_1 + 3x_2 \geq 4$, then we could replace it with the equality $x_1 + 3x_2 = 4 + s_1$.

   Rewrite the LP by taking each constraint and replacing it with another constraint involving an equality and a new variable.

3. Put this LP into *canonical form* by rearranging each inequality so that (i) there is a single variable to the left hand side, and (ii) the constants on the right hand side are all positive.

   Note: the variable on the left-hand side of each constraint should appear no where else in the linear program. If it does, replace it by what's on the right hand side of the constraint. For example, taking the constraint $x_1 + 3x_2 = 4 + s_1$ from the previous subproblem, it is natural to transform it into:

$$s_1 = -4 + x_1 + 3x_2 \ .$$

   If s_1 is a slack variable you just introduced, then it should appear only once in the linear program. On the other hand, the constant -4 is negative. To get a positive constant you could replace the constraint with

$$x_1 = 4 + s_1 - 3x_2$$

and replace any other occurence of x_1 in the linear program with $4 + s_1 - 3x_2$.

Put the constraints of your linear program into canonical form by making one or more transformations like the one above.

4. **Removing negative coefficients in the objective function**. To complete the transformation of your linear program, you need to get rid of any negative coefficients in the objective function. It's possible to do this using what's called a *pivot operation*. In a pivot operation, you choose a variable in the objective function, select a constraint containing it, and make a substitution based on that constraint. For example, if your objective was to minimize $4 - x_1 + 2x_2$, and there was a constraint

$$s_1 = 3 - x_1 - x_2 \ ,$$

then you could transform the constraint into

$$x_1 = 3 - s_1 - x_2$$

and replace all other occurrences of x_1 in the LP with $3 - s_1 - x_2$.

Now, **perform pivot operations** until all variables in the objective function have positive coefficients.

5. Solve the following minimization problem:

$$\begin{array}{ll} \text{Minimize} & 2x_1 + 2x_2 + 3x_3 \\ \text{subject to:} & x_1 + x_2 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \end{array}$$