This week, we’ve been discussing ways to classify problems according to their difficulty, using the notions of polynomial-time reductions and polynomial-time verifiers. In this lab, you’ll develop more sophisticated polynomial-time reductions using gadgets.

Below is a synopsis of relevant decision problems for this lab.

- **SAT.** The input for SAT is a set of \( n \) boolean variables \( x_1, \ldots, x_n \) and \( m \) clauses \( c_1, \ldots, c_m \), where each clause is the OR of one or more literals\(^1\) e.g. \( c_i = x_1 \lor \bar{x}_2 \lor x_3 \lor \bar{x}_{17} \). Output YES iff there is a truth assignment to \( x_1, \ldots, x_n \) that satisfies every clause.

- **3-Sat.** The input for 3-Sat is the same as for SAT, except that each clause is the OR of exactly three literals.

- **Three-Coloring.** The input for Three-Coloring is a graph \( G = (V, E) \). Output YES iff the vertices can be colored using three colors such that each edge has different-colored endpoints.

1. In the first exercise, you will reduce \( 3\text{-Sat} \leq_p \text{Three-Coloring} \). Before getting there, it will be helpful to create some interesting three-colorable graphs. In all of the following exercises, you are to create a three-colorable graph (say the colors are red, blue, green) with certain special properties. The graphs you create should include three vertices marked \( a, b, c \) but can (and often will) include other vertices. Except for the properties specified, these vertices should be unconstrained. For example, unless the problem states that e.g. \( a \) cannot be red, it must be possible to color the graph in such a way that \( a \) is red. (You may fix colors for other vertices, just not \( a, b, c \), and not in a way that constrains the colors of \( a, b, c \).)

   (a) Create a graph such that \( a, b, c \) all have different colors.

   (b) Create a graph such that \( a, b, c \) all have the same color.

   (c) Create a graph such that \( a, b, c \) do NOT all have the same color.

   (d) Create a graph such that none of \( a, b, c \) can be green.

   (e) Create a graph such that none of \( a, b, c \) are green, and they cannot all be blue.

2. Show that \( 3\text{-Sat} \leq_p \text{Three-Coloring} \). (Hint: Associate the color red with True and the color blue with False.)

3. Show that \( \text{Three-Coloring} \in \text{NP-Complete} \).

4. Show that \( \text{Sat} \leq_p 3\text{-Sat} \).

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\(^1\)A literal is either a boolean variable \( x_i \) or its negation \( \bar{x}_i \).