CS41 Homework 9

This homework is due at 10:00PM on Sunday, November 17. Write your solution using L\TeX. Submit this homework using github as a .tex file. This is a partnered homework. You should primarily be discussing problems with your homework partner.

It’s ok to discuss approaches at a high level with others. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you’ve done with a lab partner while in lab. In this case, note (in your homework submission poll file) who you’ve worked with and what parts were solved during lab.

1. Credit Networks
A credit network is a mathematical model that can capture financial relationships among banks or trust relationships between individuals. A credit network involves a set of agents $A$ (the banks or people) and credit relationships between them. Each agent $a \in A$ specifies an integer amount of credit they extend $c_{ab} \geq 0$ to each other agent $b \in A$. This credit allows agents to engage in transactions: if agent $a$ extends credit to agent $b$, then $b$ can spend that credit to obtain something of value from $a$ (banks might purchase commodities; individuals might request favors). When this happens, some of the credit is expended: if the transaction has value $v \leq c_{ab}$, then there will be $c_{ab} - v$ units of credit remaining; at the same time, agent $a$ gains $v$ units of credit from agent $b$ (equivalent to an I.O.U. indicating $b$’s obligation to later repay the debt). See Figure 1 for an example.

Agents in a credit network can also engage in remote transactions, obtaining something of value from an agent who hasn’t extended them credit directly. Here is how remote transactions work: suppose that agent $a$ extends $c_{ab}$ units of credit to agent $b$, agent $b$ extends $c_{bd}$ units of credit to agent $d$, and agent $d$ extends $c_{de}$ units of credit to agent $e$. If agent $e$ wants to make a purchase from agent $a$, they can first transact with agent $d$, using their credit from $d$ to buy some of $d$’s credit from agent $b$, then use that credit from agent $b$ to buy credit from agent $a$, and finally make a purchase from agent $a$ using this credit. See Figure 2 for an example.

![Figure 1: b buys from a, expending v units of credit.](image-url)
Figure 2: A possible sequence of remote transactions.

(a) An important query about credit networks is whether a certain transaction is feasible. In the \textsc{Transaction-Feasibility} problem we are given a credit network, two agents \(s\) and \(t\), and a transaction size \(v\). Our job is to determine whether it is possible for agent \(s\) to make a purchase from agent \(t\) of total value \(v\) in the credit network. Describe and analyze an efficient algorithm for the transaction feasibility problem.

(b) In some settings, agents might be unwilling to exchange large amounts of credit for a single transaction. This leads to a variant of the problem where we have the additional input of an exchange limit \(\ell_a\) for each agent \(a\) \(\in\ A\), and the additional constraint that for any agent \(a\) other than \(s\) and \(t\), the total amount of other agents’ credit they sell as a part of any given transaction cannot exceed \(\ell_a\). The \textsc{Exchange-Limited-Transaction-Feasibility} problem asks whether a transaction of size \(v\) between agents \(s\) and \(t\) is feasible, while respecting the exchange limits. Describe and analyze an efficient algorithm for this problem.

2. Give a polynomial-time verifier for \textsc{Subset-Sum}.

3. You are given a directed graph \(G = (V, E)\) with weights \(w_e\) on its edges \(e \in E\). The weights can be negative or positive. A simple cycle is a path \(v_1, v_2, \ldots, v_k\) of nodes in \(G\) in which \(k > 2\) and the nodes are all distinct (except that \(v_1 = v_k\)). The \textsc{Zero-Weight-Cycle} problem is to decide if there is a simple cycle in \(G\) so that the sum of the edge weights on the cycle is exactly 0. Show that \textsc{Subset-Sum} \(\leq_P \textsc{Zero-Weight-Cycle}\).

4. Extra Challenge. In a credit network, sometimes instead of setting an exchange limit, agents are allowed to charge an exchange rate. If agent \(a \in A\) charges exchange rate \(r_a\), then exchanging each unit of credit incurs a cost of \(r_a\). For example, when agent \(s\) purchases \(v\) units of agent \(b\)’s credit from agent \(a\), agent \(s\) must pay a cost of \(r_a \cdot v\).

(a) The \textsc{min-cost transaction} problem is to determine whether a transaction of size \(v\) between agents \(s\) and \(t\) is feasible, and if so, to determine the set of credit exchanges that will complete the transaction at the lowest total cost (summed over all nodes where credit is exchanged). Describe and analyze an efficient algorithm for the \textsc{min-cost transaction} problem.

(b) In a credit network, it is natural to denominate costs not in a common currency, but in units of the credit being exchanged. In the \textsc{max-credit transaction} problem, exchange rates must be paid with additional credit, so purchasing \(v\) units of credit from agent \(a\) costs \((1 + r_a)\) units of agent \(a\)’s credit. In this case, we must determine whether a transaction is feasible, and if so, maximize the total direct credit available to the purchasing agent \(t\) after the transaction: \(\max \sum_{a \in A} c_{at}\). Describe and analyze an efficient algorithm for the \textsc{max-credit transaction} problem.