CS41 Homework 2

This homework is due at 11:59PM on Sunday, September 16. Write your solution using LaTeX. Submit this homework using **github** as a file called **hw2.tex**. This is an individual homework. It's ok to discuss approaches at a high level. In fact, we encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you've done with a lab partner *while in lab*. In this case, note who you've worked with and what parts were solved during lab.

1. **Stable Matching Runtime.** We showed in class (at least after Friday's class) that the Gale-Shapely Algorithm for stable matching terminates after at most n^2 iterations of the while loop.

   (a) For two sets A and B of size n, can a particular list of rankings actually result in a quadratic number of iterations? If so, describe what the rankings would look like. If not, argue why no set of rankings would ever result in a quadratic number of iterations.
   Note: the algorithm need not take exactly n^2 iterations, but *asymptotically* n^2 iterations, meaning $0.1n^2$ would be sufficient to show your claim.

   (b) Can a particular set of rankings result in strictly less than a quadratic number of iterations? Can you design an input that requires $O(n)$ iterations? If so, describe the structure of this input. If not, argue why this is not possible.

   (c) Finally, can you design an input that takes fewer than n iterations? Why or why not?

2. **Practice with proofs.**

   (a) Sometimes a proof which follows the correct structure can have a subtle problem. Consider the following proof.

   Claim 1. *All dogs are the same color.*

   Proof. (by induction)

   - base case: Let's say we have just $x = 1$ dog. Then clearly all the dogs we're considering are the same color.
   - inductive hypothesis: Assume that if we have $x = k$ dogs in a group, they are all the same color, for some arbitrary constant k.
   - inductive step: Let's say we have $x = k + 1$ dogs in a group. We want to show that they are all the same color.
     Let's consider the group if we take out one dog ("Jayjay"). Now we have a group of k dogs, so by the inductive hypothesis, they are all the same color.
     Now add Jayjay back in, and let's take out a different dog ("Hairy Potter"). Now again we have a group of k dogs, so by the inductive hypothesis, they are all the same color.
     Since the group with Potter but not Jayjay is all the same color, and the group with Jayjay but not Potter is all the same color, it must be that Jayjay and Potter are the same color. Therefore all $k + 1$ dogs in the group are the same color.

Therefore, by induction, all dogs are the same color. □

It seems like something is probably wrong with this argument, since dogs exist in many different colors. Explain what the issue is with this proof.

(b) Prove or disprove the following claim:

Claim 2. *All CS professors wear an outfit consisting of a plaid button-down shirt and khakis when they teach.*

3. **College Choice.** A group of n high school seniors $S = \{s_1, s_2, s_3, \ldots, s_n\}$ are touring a set of colleges $C == \{c_1, \ldots, c_n\}$ over the course of $m \geq n$ days this autumn. Each student s_j has an itinerary where he/she decides to visit one college per day (or maybe take a day off if $m > n$). However, these students are fiercely independent and prefer not to share college campus tours with other students. Furthermore, each student is looking for one college to call their own. Each student s would like to choose a particular day d_s and stay at his/her current college admissions office for the remaining $m - d_s$ days of the autumn (laying claim to 100% of the admissions officer's time). Of course, this means that no other students can visit c_s after day d_s, since students don't like sharing.

   Show that no matter what the students' itineraries are, it is possible to assign each student s a unique college c_s, such that when s arrives at c_s according to the itinerary for s, all other students s' have either stopped touring colleges themselves, or s' will not visit c_s after s arrives at c_s. Describe an algorithm to find this *matching*.

   Hint: The input is somewhat like the input to stable matching, but at least one piece is missing. Find a clever way to construct the missing piece(s), run stable matching, and show that the final result solves this problem.

4. **(extra challenge problem)** In class, we discussed a version of the stable matching problem where we want to match n doctors to n hospitals. In this problem, we discuss the homogeneous version. The input is a set of students $A = \{s_1, \ldots, s_{2n}\}$ of size $2n$. Each student ranks the others in order of preference. A homework partner assignment of students into partners $M = \{(i, j)\}$ is a matching; it is unstable if there exists $(i, j), (i', j') \in M$ such that i prefers j' to j and that j' prefers i to i'. It is stable if it is a perfect matching and there are no instabilities.

   (a) Does a stable homework partner assignment always exist? Prove that such an assigment must always exist, or give an example where no stable assignment occurs. (Remember, you must have $2n$ students.)

   (b) Design and analyze an efficient algorithm that either returns a stable matching for homework partners or outputs that no such matching exists.