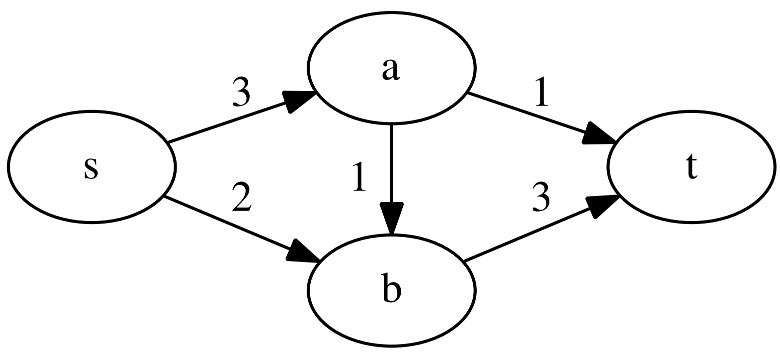
Linear programming and duality

A reminder of some linear programming vocabulary:

- A linear program in **canonical form** is written as: $\min \vec{c}^t \cdot \vec{x}$ subject to **constraints** $A \cdot \vec{x} \geq \vec{b}$ and $\vec{x} \geq 0$, where the matrix A and vectors $\vec{c}$ and $\vec{b}$ are all constants.
- The **objective function** is the thing we are trying to minimize/maximize in a linear program ($\vec{c}^t \cdot \vec{x}$ in canonical form).
- A **feasible solution** to a linear program is an assignment of values to the variables $\vec{x}$ which satisfies all the constraints.

LPs for flow networks. Last week we saw that the maximum flow in a network corresponds to the minimum capacity of any s-t cut in that network.

Consider the following example network with edge capacities.



We saw that we could write the maximum flow from s to t as the solution to a linear program:

$$
\begin{array}{rllllll}
\max & f_{sa} & +f_{sb} & & & & \\
\text{subject to:} & f_{sa} & & & & & \leq 3 \\
 & & f_{sb} & & & & \leq 2 \\
 & & & f_{ab} & & & \leq 1 \\
 & & & & f_{at} & & \leq 1 \\
 & & & & & f_{bt} & \leq 3 \\
 & f_{sa} & & -f_{ab} & -f_{at} & & = 0 \\
 & & f_{sa} & +f_{sb} & & -f_{bt} & = 0 \\
 & & & & & & f \geq 0
\end{array}
$$

The interpretation of these variables was that each edge (u, v) got a corresponding f_{uv} variable which expressed how much flow was assigned to that edge. The edge capacity limit is enforced by the inequality constraints. The conservation of flow requirement is enforced by the equality constraints. Thus a feasible solution to this linear program represents a legal s-t flow on the network, and the objective function we are maximizing is exactly the value of the flow. (In class we rewrote this linear program to be in canonical form.)

It's reasonably clear that this linear program encodes the maximum flow. But what about the minimum cut?

Consider the following linear program:

$$\begin{array}{rlllllll}
\min & 3y_{sa} & +2y_{sb} & +y_{ab} & +y_{at} & +3y_{bt} & & \\
\text{subject to:} & y_{sa} & & & & & +u_a & & \geq 1 \\
& & y_{sb} & & & & & +u_b & \geq 1 \\
& & & y_{ab} & & & -u_a & +u_b & \geq 0 \\
& & & & y_{at} & & -u_a & & \geq 0 \\
& & & & & y_{bt} & & -u_b & \geq 0 \\
& & & & & & & & \vec{y} \geq 0
\end{array}$$

This linear program describes the minimum cut problem! Suppose that the u_a variable is 1 if a is in the cut with s, and 0 otherwise, and similarly u_b. For each edge (u, v), the variable y_{uv} is 1 if the edge contributos to the cut capacity, and 0 otherwise.

The constraints enforce these requirements on the variables. For example, the first constraint states that "if a is not on the same side of the cut as s, then (s, a) must be added to the cut capacity."

Although the y and u variables are free to take values larger than one, the objective function is a minimization, so it will force them to be as small as possible.

These two linear programs are remarkably similar. Indeed, if we write the flow-maximizing LP as vectors and matrices, we have:

$$\begin{array}{rl}
\min & \vec{c}^t \cdot \vec{x} \\
\text{subject to:} & A \cdot \vec{x} \geq \vec{b} \\
& \vec{x} \geq 0
\end{array}$$

Call this the **primal** linear program.

Now, *using the same constants* A, $\vec{c}$, and $\vec{b}$, the cut-capacity-minimizing LP can be written:

$$\begin{array}{rl}
\max & \vec{b}^t \cdot \vec{y} \\
\text{subject to:} & A^t \cdot \vec{y} \leq \vec{c} \\
& \vec{y} \geq 0
\end{array}$$

This is the **dual** linear program. Each variable of the primal corresponds to a constraint of the dual, and vice-versa. The equality constraints correspond to unrestricted variables (the us), and the inequality constraints correspond to restricted variables. Minimization becomes maximization, the matrices are transposes of each other, and the roles of the objective function and the constant bounds are interchanged.

Every LP has a dual formed in this way. (The particular details are not necessary for this class. There are entire courses taught on linear programming!) As the terminology suggests, the dual of a dual is the primal again.

By the theorem from class, we know that the maximum flow is equal to the capacity of the minimum cut. In fact, this is true *in general* for dual linear programs.

Theorem 1. *If a linear program has a bounded optimum, then so does its dual, and the two optimal values are equal.*