CS 31: Introduction to Computer Systems

03: Binary Arithmetic

January 28

SWARTHMORE COLLEGE

Clicker Question:
Have you registered your clicker?

A. Yes
B. No
C. What’s a clicker?

Check your frequency:

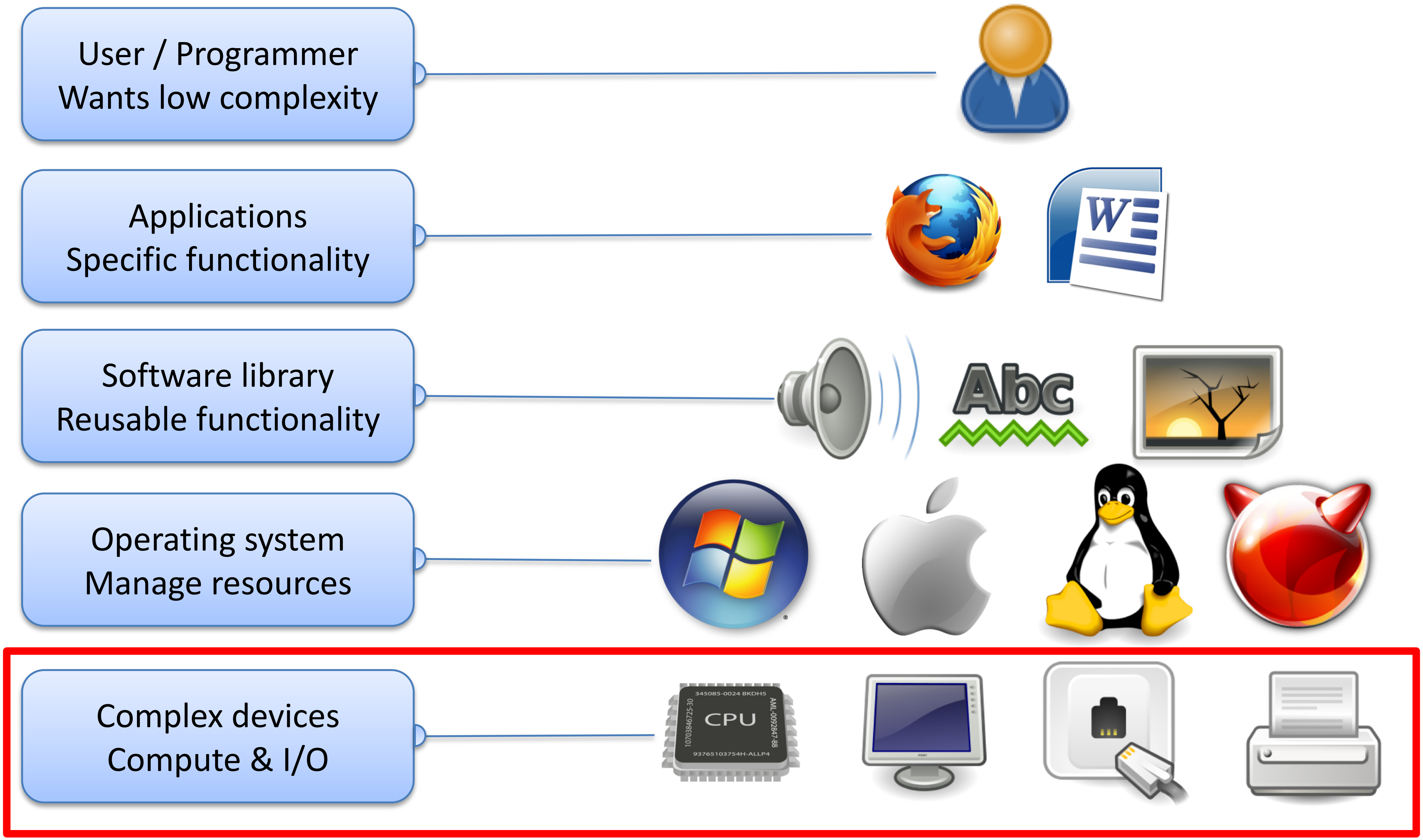
- Iclicker2: frequency AA
- Iclicker+: green light next to selection

For new devices this should be okay,
For used you may need to reset frequency

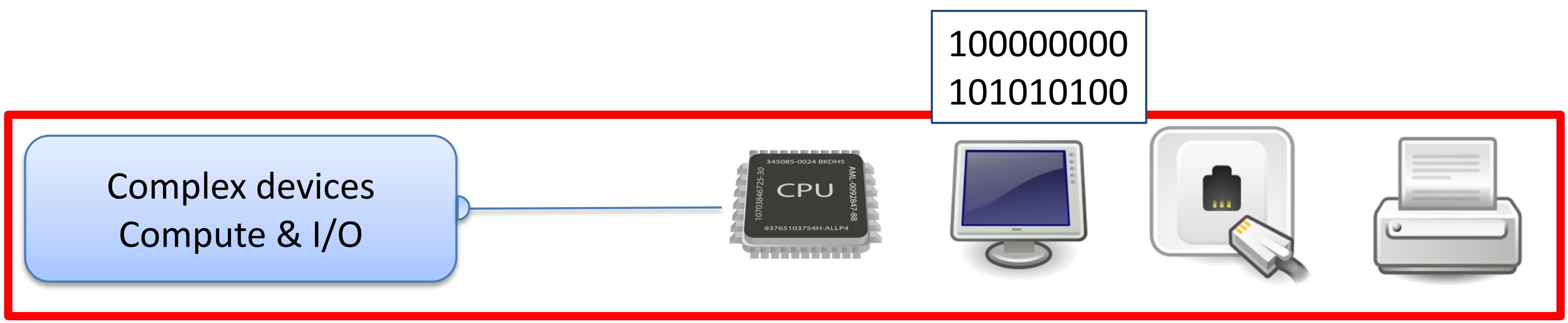
Reset:
1. hold down power button until blue light flashes (2secs)
2. Press the frequency code: AA vote status light will indicate success

Reading Quiz

Abstraction



Abstraction



Today

- Binary Arithmetic
  - Unsigned addition
  - Subtraction
- Representation
  - Signed magnitude
  - Two’s complement
  - Signed overflow
- Bit operations

Last Class: Binary Digits: (BITS)

	7 6 5 4 3 2 1 0	
Most significant bit →	10001111	← Least significant bit

Representation: $1 \times 2^7 + 0 \times 2^6 \ldots\ldots + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

10001111 = 143

one byte is the smallest addressable unit - contains 8 bits

Last Class: Unsigned Integers

- Suppose we had one byte
  - Can represent 2^8 (256) values
  - If unsigned (strictly non-negative): 0 – 255

Last Class: Unsigned Arithmetic (one byte)

- one byte
  - 2^8 (256) values
  - unsigned : 0 – 255

0 = 00000000

+1

1 = 00000001

2 = 00000010

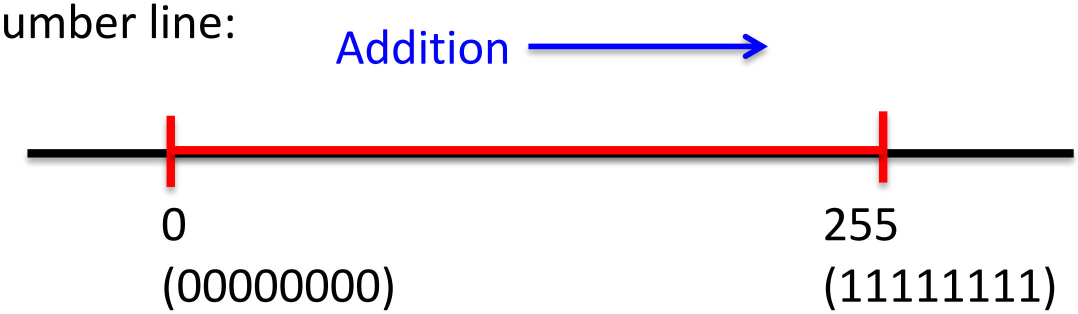
...

254 = 11111110

+1

255 = 11111111

Number line:



255 =
$1*2^7 + \ldots + 1*2^0$

Last Class: Unsigned Arithmetic (one byte)

- one byte
  - 2^8 (256) values
  - unsigned : 0 – 255

we cannot represent an infinite number of values in a finite storage space

252 = 11111100

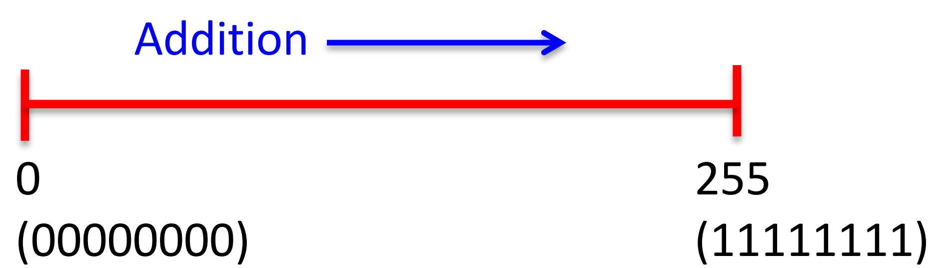
+1

253 = 11111101

254 = 11111110

255 = 11111111

+1



What if we add one more?

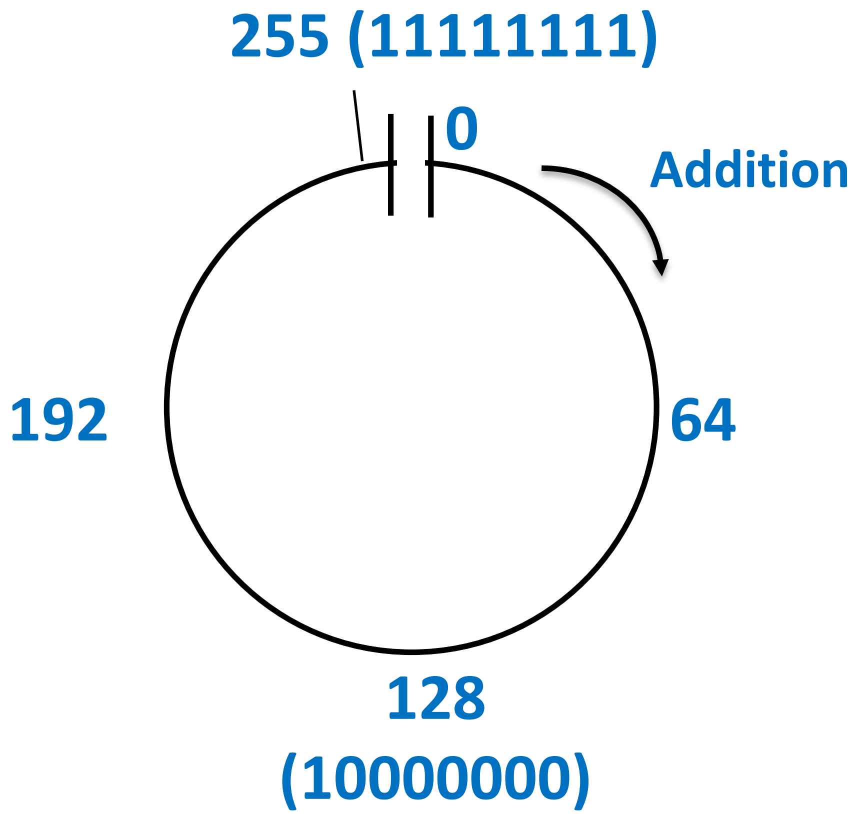
255 + 1 is ?

Car odometer "rolls over".

Last Class: Arithmetic and Fixed Storage

- Fixed Storage: finite set of values
  - 1 byte: 2^8 (256) values
  - unsigned values: 0 – 255
- Yields Modular Arithmetic
  - All operations are % 256

(eg) 255 + 4 = 259 % 256 = 3



Modular arithmetic: Here, all values are modulo 256.

Last Class: Unsigned Addition (4-bit)

Addition works like grade school addition:

```
   1
   0110       6
+  0100   +   4
--------  -----
   1010      10
```

Four bits give us range: 0 - 15

Last Class: Unsigned Addition (4-bit)

Addition works like grade school addition:

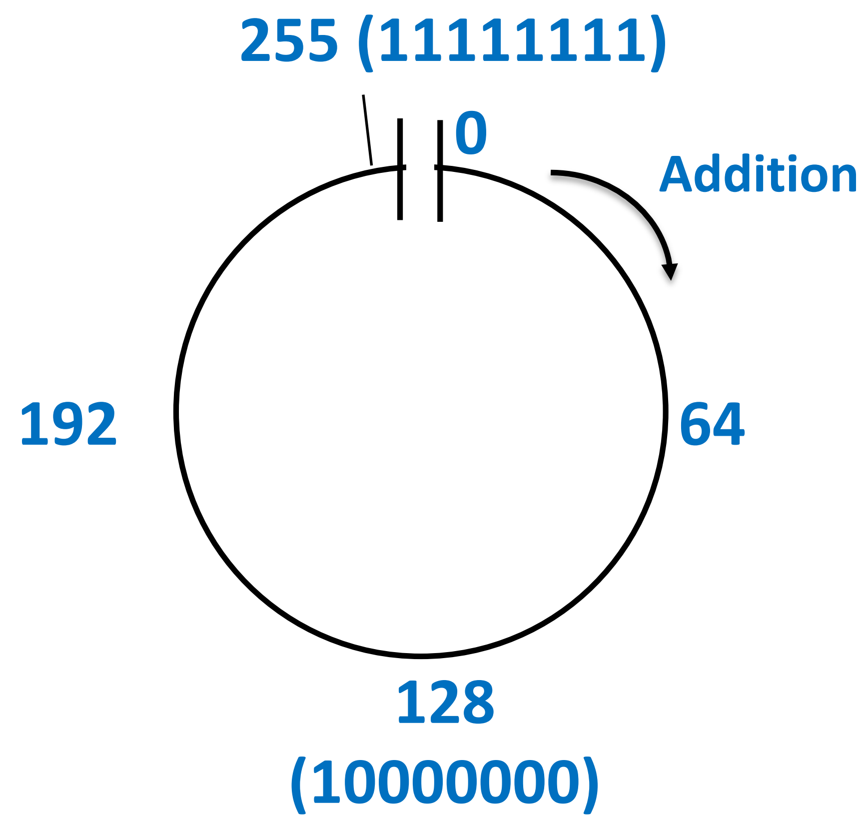
```
   1
   0110     6           1100     12
+  0100   + 4        +  1010    +10
---------------      ----------------
   1010    10        1  0110      6
                     ^carry out
```

Four bits give us range: 0 - 15

Overflow!

Last Class: Arithmetic and Fixed Storage

- Fixed Storage: finite set of values
  - 1 byte: 2^8 (256) values
  - unsigned values: 0 – 255



Not Used: Signed Magnitude Representation (for 4 bit values)

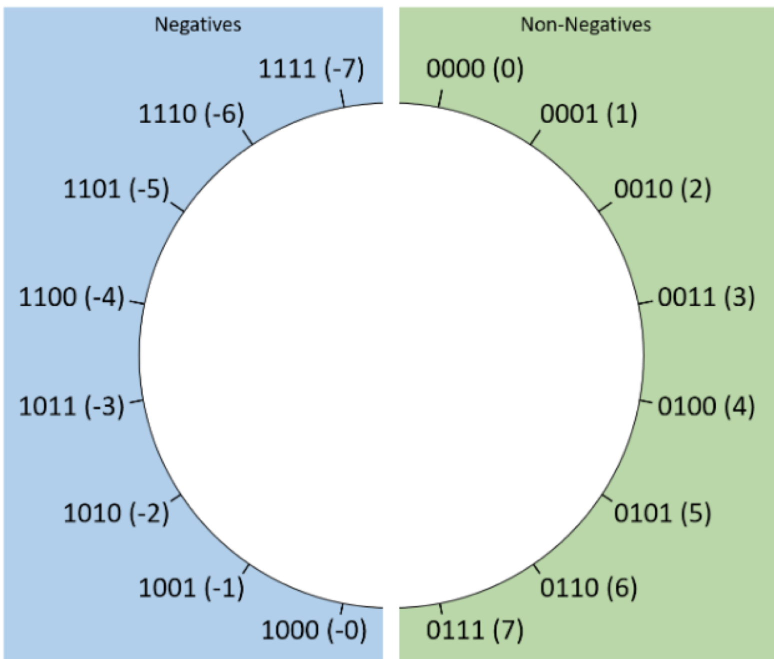


Figure 1. A logical layout of signed magnitude values for bit sequences of length four.

One bit (usually left-most) signals:

- 0 for positive
- 1 for negative

For one byte:

1 = 00000001

-1 = 10000001

Pros: Negation (negative value of a number) is very simple!

For one byte:

0 = 00000000

-0? = 10000000

Major con: Two ways to represent zero.

Used Today: Two's Complement Representation (for four bit values)

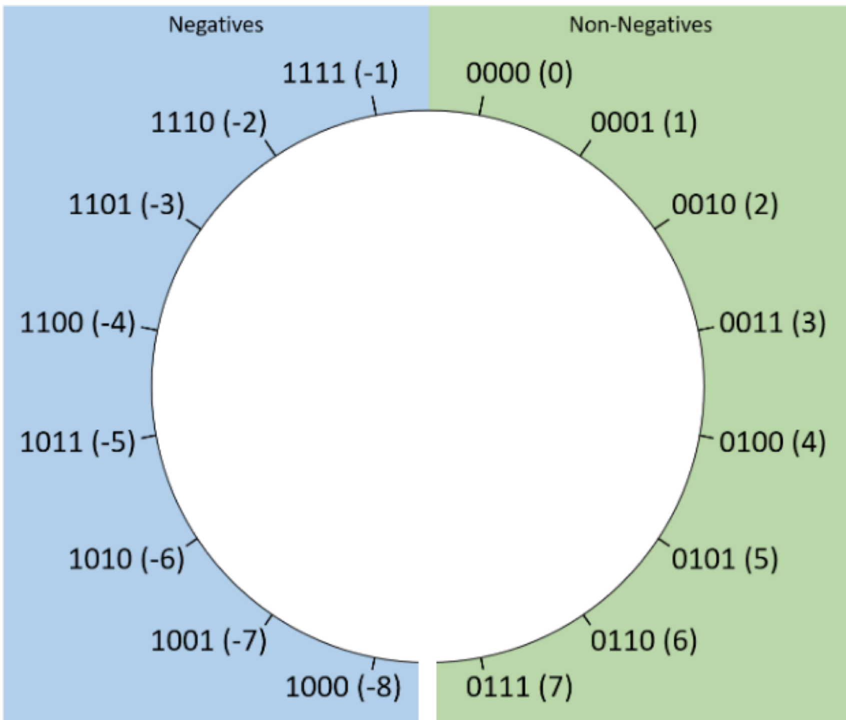
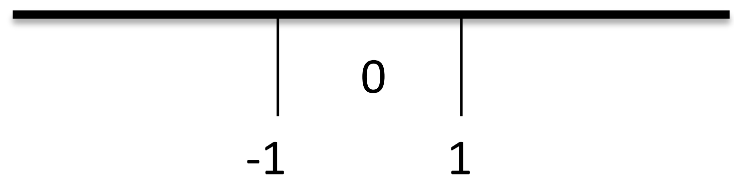


Figure 2. A logical layout of two's complement values for bit sequences of length four.

- Borrow nice property from number line:



Only one instance of zero!
Implies: -1 and 1 on either side of it.

For an 8 bit range we can express 256 unique values:

- 128 non-negative values (0 to 127)
- 128 negative values (-1 to -128)

Two's Complement

- Only one value for zero
- With N bits, can represent the range:
  - -2^{N-1} to $2^{N-1} - 1$
- Most significant bit still designates
  - 0: positive
  - 1: negative

- Negating a value is slightly more complicated:

  1 = 00000001,   -1 = 11111111

From now on, unless we explicitly say otherwise, we'll assume all integers are stored using two's complement! This is the standard!

Two's Complement

- Each two's complement number is now:

$$[-2^{n-1}*d_{n-1}] \quad + \quad [2^{n-2}*d_{n-2}] \quad +\ldots+ \quad [2^{1}*d_{1}] \quad + \quad [2^{0}*d_{0}]$$

Note the negative sign on just the most significant bit. This is why first bit tells us whether the value is negative vs. positive.

If we interpret 11001 as a two's complement number, what is the value in decimal?

Each two's complement number is now:

$[-2^{n-1} * d_{n-1}] \quad + \quad [2^{n-2} * d_{n-2}] \quad + \ldots + \quad [2^{1} * d_{1}] \quad + \quad [2^{0} * d_{0}]$

A. -2

B. -7

C. -9

D. -25

If we interpret 11001 as a two's complement number, what is the value in decimal?

Each two's complement number is now:

$[-2^{n-1} * d_{n-1}] \quad + \quad [2^{n-2} * d_{n-2}] \quad + \ldots + \quad [2^{1} * d_{1}] \quad + \quad [2^{0} * d_{0}]$

A. -2

B. -7 $\qquad -16 + 8 + 1 = -7$

C. -9

D. -25

"If we interpret…"

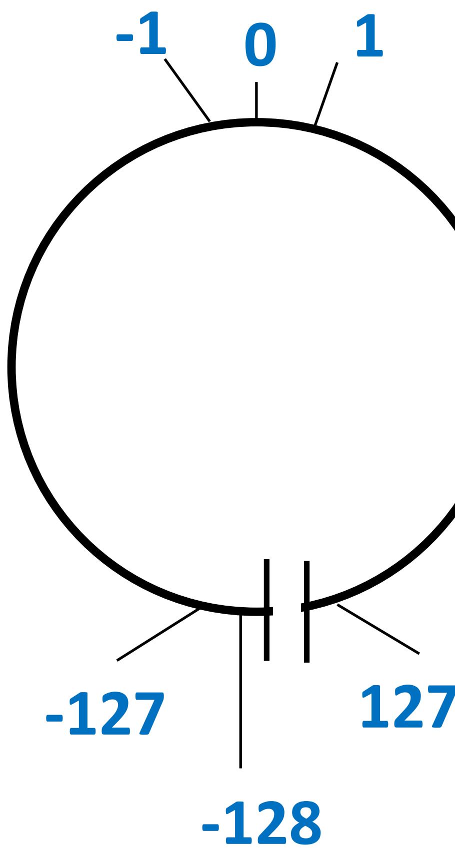
What is the decimal value of 1100?

- …as unsigned, 4-bit value: 12 (%u)
- …as signed (two's comp), 4-bit value: -4 (%d)

- …as an 8-bit value: 12. (i.e., **0000** 1100)

Two's Complement Negation

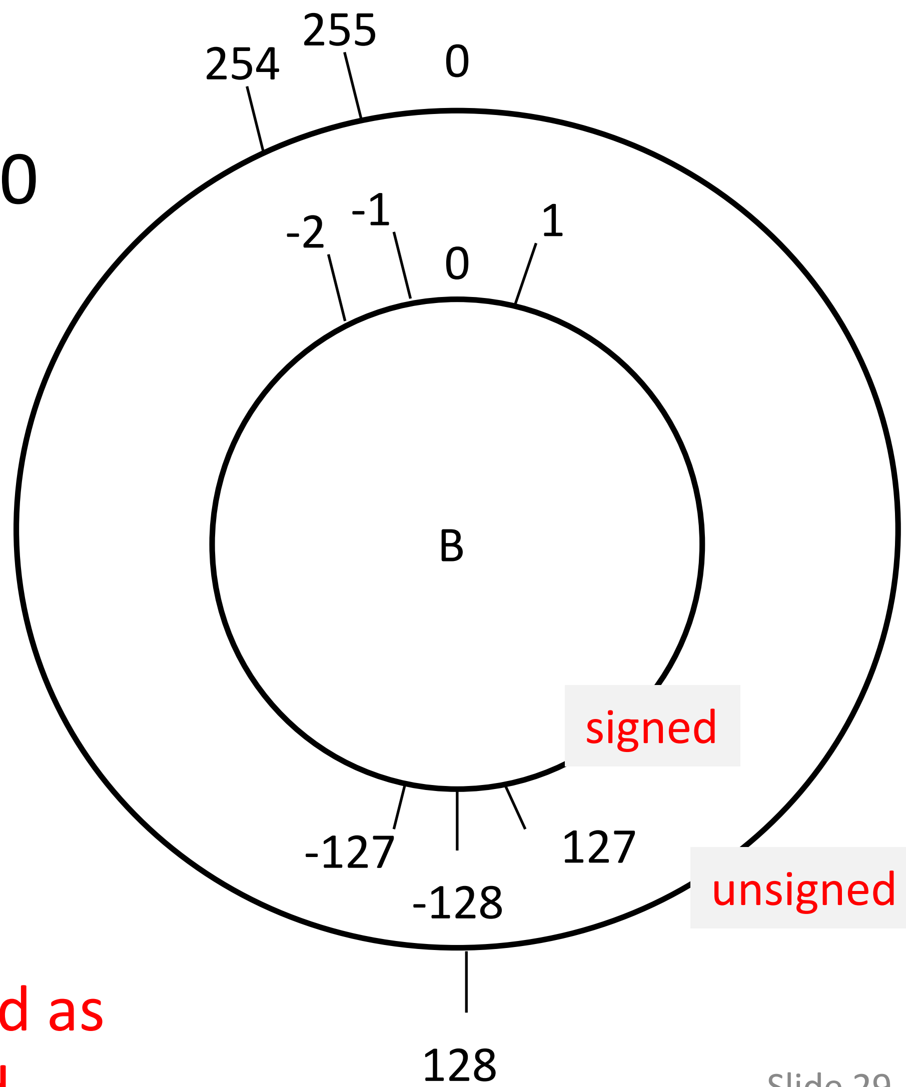
- To negate a value x, we want to find y such that x + y = 0.
- For N bits, $y = 2^N - x$



Negation Example (8 bits)

- For N bits, $y = 2^N - x$
- Negate the value (2) 00000010
- $2^8 - 2 = 256 - 2 = 254$

- Our wheel only goes to 127!
  – Put -2 where 254 would be if wheel was unsigned.
  – 254 in binary is 11111110



Given 11111110, it's 254 if interpreted as <u>unsigned</u> and -2 interpreted as <u>signed</u>.

Negation Shortcut

- A much easier, faster way to negate:
  - Flip the bits (0's become 1's, 1's become 0's)
  - Add 1
- Negate 00101110 (46)
- Formally:
  - 2^8 - 46 = 256 - 46 = 210
  - 210 in binary is 11010010

46:	00101110
Flip the bits:	11010001
Add 1	+ 1
-46:	11010010

Negation Two Ways

4 bit Examples			
x	-x	2^4 - x	Bit flip + 1
0000	0000	10000 – 0000 = 0000	1111 + 1 = 0000
0001	1111	10000 – 0001 = 1111	1110 + 1 = 1111
0010	110	10000 – 0010 = 1110	1101 + 1 = 1110
0111	1001	10000 – 0111 = 1001	1000 + 1 = 1001

Decimal to Two's Complement with 8 bit values (high-order bit is the sign bit)

for positive values, use same algorithm as for unsigned

- `(E.g.)6   6 - 4 = 2` $(4{:}2^2)$
- `2 - 2 = 0` $(2{:}2^1)$`:   00000110`

for negative values:

- convert negation (positive) to binary
- then negate binary to get negative

`E.g.: -3`

- `3: 00000011`
- `negate: 11111100+1 = 11111101 = -3`

Decimal to Two's Complement with 8 bit values (high-order bit is the sign bit)

for negative values:

- convert negation (positive) to binary
- then negate binary to get negative

Try converting -7 to Two's Complement representation

A. 11111001
B. 00000111
C. 11111000
D. 11110011

Decimal to Two's Complement with 8 bit values (high-order bit is the sign bit)

for negative values:

- convert negation (positive) to binary
- then negate binary to get negative

Try converting -7 to Two's Complement representation

A. 11111001
B. 00000111
C. 11111000
D. 11110011

-7 = (1) 7: 00000111
(2) negate: 11111000 + 1 = 11111001

Addition & Subtraction for Integers

- Addition is the same as for unsigned
  - One exception: different rules for overflow
  - Can use the same hardware for both
- Subtraction is the same operation as addition
  - Just need to negate the second operand…
- 6 - 7 = 6 + (-7) = 6 + (~7 + 1)
  - ~7 is shorthand for "flip the bits of 7"

Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1

input 1 ------------------------------->

input 2 --> $\boxed{\text{possible bit flipper}}$ --> ADD CIRCUIT ---> result

possible +1 input-------->

Let's call this possible +1 input: "Carry in"

(0: on add, 1: on subtract)

4-bit signed Examples:

Subtraction via Addition:

– a-b is same as a + ~b + 1

Subtraction: flip bits and add 1

```
3 -  6 =    0011
            1001      (6: 0110  ~6: 1001)
          +    1
            ----
            1101 = -3
```

Addition: don't flip bits or add 1

```
3 + -6 =    0011
          + 1010
            ----
            1101 = -3
```

Signed & Unsigned 4-bit Subtraction:

Unsigned subtraction: flip bits and add 1

```
13 -  1 =
```

Signed subtraction: flip bits and add 1

```
-3 - 1 =
```

A. 1100 & 1100
B. 1100 & 1010
C. 1010 & 1010
D. 1001 & 1100

Signed & Unsigned 4-bit Subtraction:

Unsigned subtraction: flip bits and add 1

```
13 -  1 =  1101
           1110      (1: 0001  ~1: 1110)
        +  ____1
       1   1100 = 12
```

Signed subtraction: flip bits and add 1

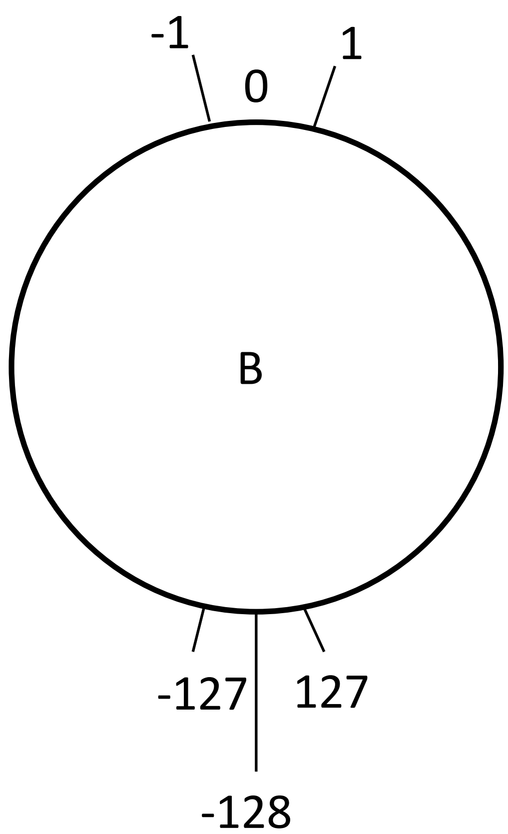
```
-3 - 1 =   1101
           1110
        +  ____1
       1   1100 = -4
```

By switching to two's complement, have we solved this value "rolling over" (overflow) problem?

A. Yes, it's gone.

B. Nope, it's still there.

C. It's even worse now.



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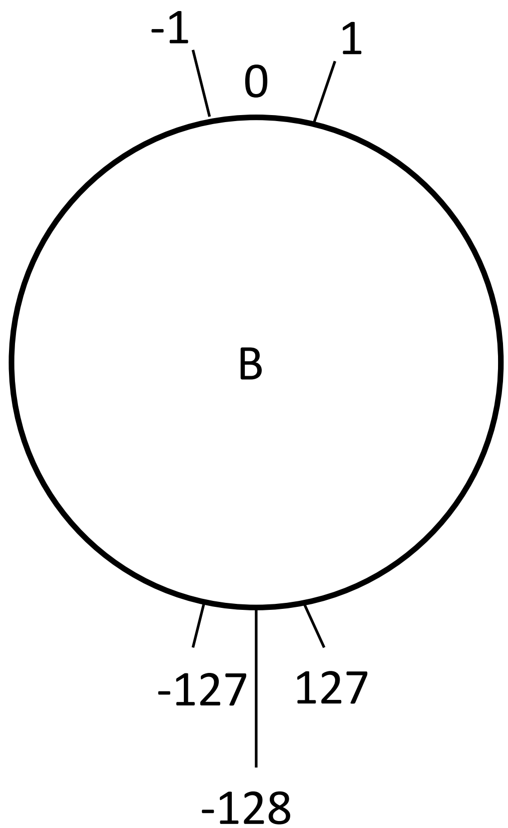
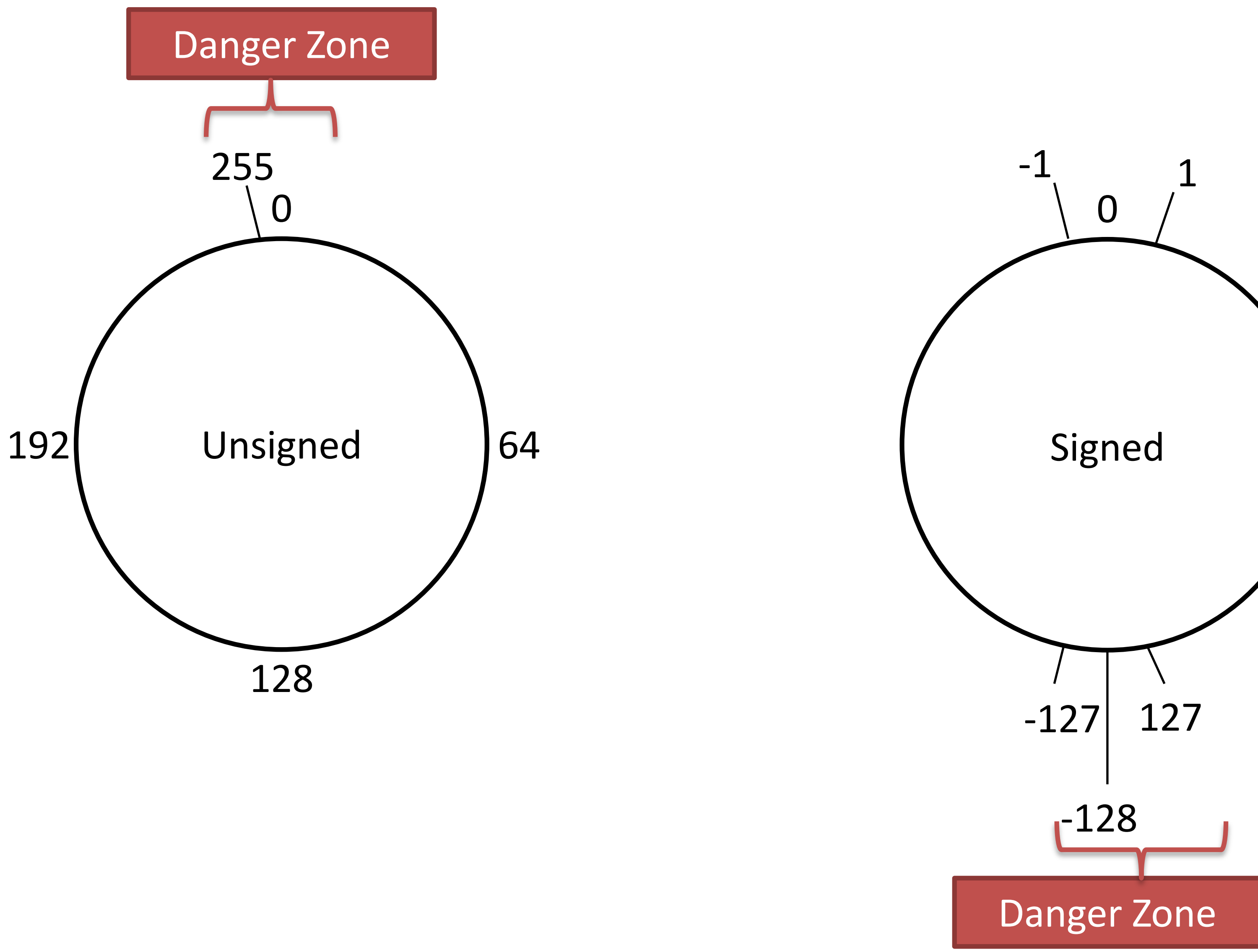


This is an issue we need to be aware of when adding and subtracting!

Overflow, Revisited

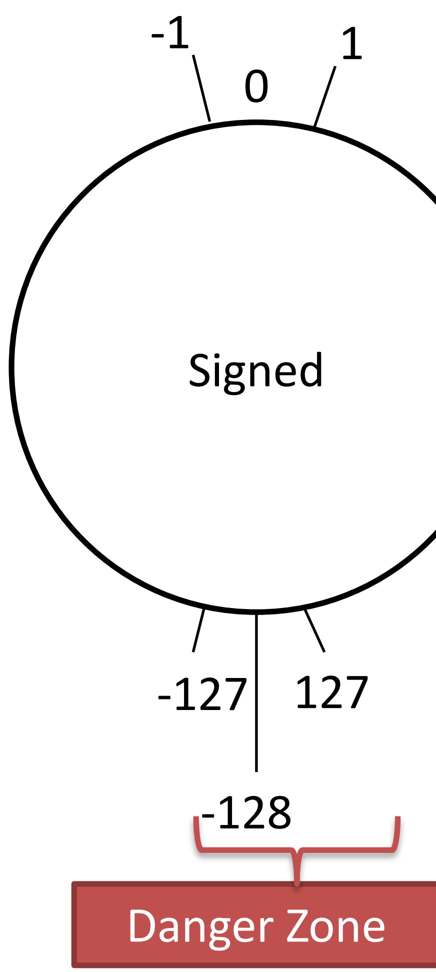


If we add a positive number and a negative number, will we have overflow? (Assume they are the same # of bits)

A. Always

B. Sometimes

C. Never



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A. Always

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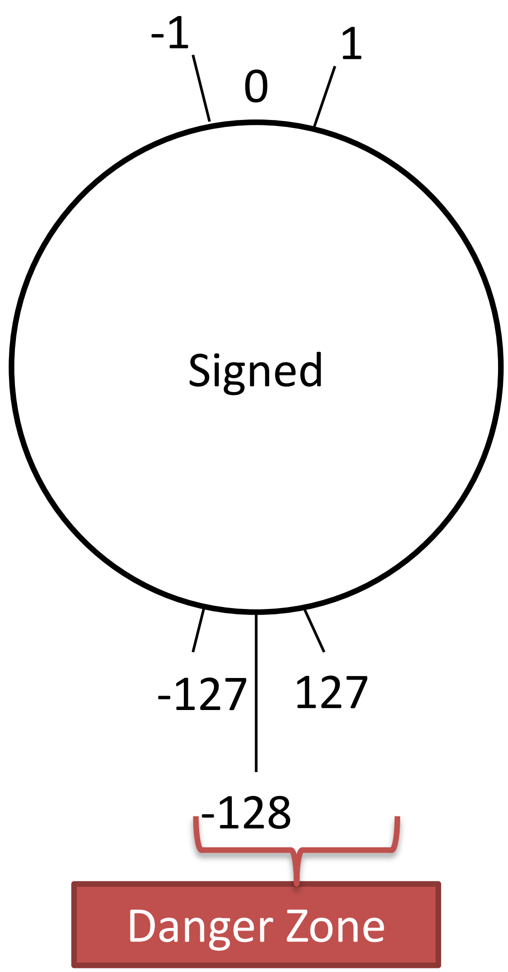


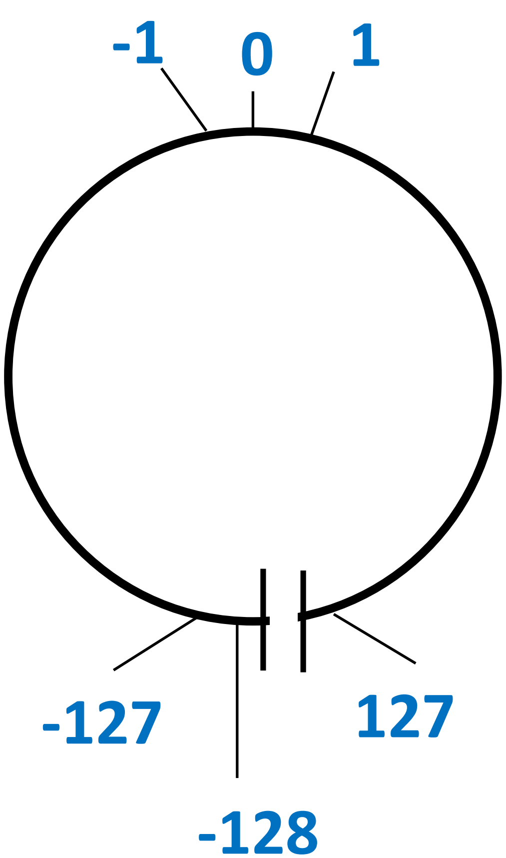
Signed (Two's Complement) Overflow For Addition

- **Addition Overflow**: IFF the sign bits of operands are the same, but the sign bit of result is different.
- Not enough bits to store result!

sign of operands = sign of result

no overflow

```
 3+4=7        -2+-3=-5
 0011          1110
+0100         +1101
 0111        1  1011
```



Signed (Two's Complement) Overflow For <u>Addition</u>

- **<u>Addition Overflow</u>**: IFF the sign bits of <u>operands are the same,</u> but the sign bit of <u>result is different</u>.
- Not enough bits to store result!

sign of operands = sign of result

no overflow

3+4=7	-2+-3=-5
0011	1110
+0100	+1101
0111	1 1011

sign of operands ≠ sign of result

overflow

4+7=11	-6-8=-14
0100	1010
+0111	+1000
1011	1 0010

Signed (Two's Complement) Overflow For Subtraction

Subtraction Overflow Two Rules:

- **Rule 1:**

  Minuend | Subtrahend | Result

  - Positive operand - Negative operand = Positive Result: No Overflow
  - Positive operand - Negative operand = Negative Result: Overflow
  - **Intuition:** We know a positive – negative is equivalent to a positive + positive. If this sum does not result in a positive value we have an overflow

- **Rule 2:**

  Minuend | Subtrahend | Result

  - Negative operand - Positive operand = Negative Result: No Overflow
  - Negative operand - Positive operand = Positive Result: Overflow
  - **Intuition:** We know a negative – positive number is equivalent to a negative + negative number. If this sum does not result in a negative value we have an overflow

Signed (Two's Complement) Overflow For Subtraction

Subtraction Overflow Rules Summarized:

- IFF the sign bits of the subtraction operands are different, and the sign bit of the Result and Subtrahend are the same as shown below:
  - Minuend - Subtrahend = Result
  - If positive – negative = negative (overflow)
  - If negative – positive = positive (overflow)

Signed (Two's Complement) Overflow For Subtraction

Subtraction Overflow Two Rules:

– **Rule 1:**

- Positive operand - Negative operand = Positive Result: No Overflow
- Positive operand - Negative operand = Negative Result: Overflow

Subtrahend and Result have **different sign bits**

no overflow

```
2-(-3)=5          3-(-4)=7
  0010              0011
 -1110             -1100

  0010              0011
 +0011             +0100
  0101              0111
```

Subtrahend and Result have the **same sign bits**

overflow

```
2-(-6)=8          3-(-7)=10
  0010              0011
 -1010             -1001

  0010              0011
 +0110             +0111
  1000 (-8)         1010 (-6)
```

Signed (Two's Complement) Overflow For Subtraction

Subtraction Overflow Two Rules:

– **Rule 2:**

- Negative operand - Positive operand = Negative Result: No Overflow
- Negative operand - Positive operand = Positive Result: Overflow

Subtrahend and Result have **different sign bits**

no overflow

```
-2-(3)=-5        -3-(4)=-7
  1110             1101
 -0011            -0100

  1110             1101
 +1101            +1100
1 1011(-5)       1 1001(-7)
```

Subtrahend and Result have the **same sign bits**

overflow

```
-2-(7)=-9        -4-(7)=-11
  1110             1100
 -0111            -0111

  1110             1100
 +1001            +0111
1 0111(7)        1 0011(-6)
```

Overflow Rules

- Signed (Two's Complement):
  - Addition:
    - The sign bits of operands are the same, but the sign bit of result is different.
  - Subtraction:
    - First compute the following: if the sign bits of the subtraction operands are different, and the sign bit of the result and subtrahend are the same. (minuend-subtrahend – result)
    - then, turn into an Addition operation

- Can we formalize unsigned overflow?
  - Need to include subtraction too, skipped it before.

Recall Subtraction Hardware

<u>Negate and add 1 to second operand:</u>

Can use the same circuit for add and subtract:

6 - 7 == 6 + ~7 + 1

input 1 ------------------------------->

input 2 --> possible bit flipper --> ADD CIRCUIT ---> result

possible +1 input-------->

Let's call this possible +1 input: "Carry in"

(0: on add, 1: on subtract)

How many of these unsigned operations have overflowed?

4 bit unsigned values (range 0 to 15):

```
                                carry-in carry-out
                                       ↓      ↓
```
Addition (carry-in = 0)

```
9 + 11  =   1001 + 1011 + 0 =  1  0100
9 +  6  =   1001 + 0110 + 0 =  0  1111
3 +  6  =   0011 + 0110 + 0 =  0  1001
```

Subtraction (carry-in = 1)

```
                    (-3)
6 -  3  =   0110 + 1100 + 1  = 1  0011
3 -  6  =   0011 + 1010 + 1  = 0  1101
                    (-6)
```

A. 1
B. 2
C. 3
D. 4
E. 5

How many of these unsigned operations have overflowed?

4 bit unsigned values (range 0 to 15):

```
                                        carry-in  carry-out
Addition (carry-in = 0)                    ↓       ↓
  9 + 11  =   1001 + 1011 + 0 =  1   0100 =  4
  9 +  6  =   1001 + 0110 + 0 =  0   1111 = 15
  3 +  6  =   0011 + 0110 + 0 =  0   1001 =  9

Subtraction (carry-in = 1)
                     (-3)
  6 -  3  =   0110 + 1100 + 1  = 1   0011 =  3
  3 -  6  =   0011 + 1010 + 1  = 0   1101 = 13
                     (-6)
```

A. 1
B. 2
C. 3
D. 4
E. 5

Pattern?

How many of these unsigned operations have overflowed?

4 bit unsigned values (range 0 to 15):

```
                                       carry-in carry-out
                                              ↓      ↓
Addition (carry-in = 0)
   9 + 11  =   1001 + 1011 + 0 =  1  0100 =  4
   9 +  6  =   1001 + 0110 + 0 =  0  1111 = 15
   3 +  6  =   0011 + 0110 + 0 =  0  1001 =  9

                             (-3)
Subtraction (carry-in = 1)
   6 -  3  =   0110 + 1100 + 1  = 1  0011 =  3
   3 -  6  =   0011 + 1010 + 1  = 0  1101 = 13
                             (-6)
```

A. 1
B. 2
C. 3
D. 4
E. 5

Pattern?

Overflow Rule Summary

- Signed overflow:
  - The sign bits of operands are the same, but the sign bit of result is different.

- Unsigned: overflow
  - The carry-in bit is different from the carry-out.

C_{in}	C_{out}	C_{in} XOR C_{out}
0	0	0
0	**1**	**1**
1	**0**	**1**
1	1	0

So far, all arithmetic on values that were the same size. What if they're different?

Sign Extension

- When combining signed values of different sizes, expand the smaller to equivalent larger size:

```
char y=2, x=-13;
short z = 10;
```

```
z = z + y;

0000000000001010
+        00000010
0000000000000010
```

```
z = z + x;

0000000000000101
+        11110011
1111111111110011
```

Fill in high-order bits with sign-bit value to get same numeric value in larger number of bytes.

Let's verify that this works

4-bit signed value, sign extend to 8-bits, is it the same value?

0111 ---> 0000 0111 obviously still 7

1010 ----> 1111 1010 is this still -6?

-128 + 64 + 32 + 16 + 8 + 0 + 2 + 0 = -6 yes!

Operations on Bits

- For these, doesn't matter how the bits are interpreted (signed vs. unsigned)

- Bit-wise operators (AND, OR, NOT, XOR)

- Bit shifting

Bit-wise Operators

- bit operands, bit result (interpret as you please)

& (AND) | (OR) ~(NOT) ^(XOR)

A	B	A & B	A | B	~A	A ^ B
0	0	0	0	1	0
0	1	0	1	1	1
1	0	0	1	0	1
1	1	1	1	0	0

```
  01010101       01101010       10101010     ~10101111
| 00100001     & 10111011     ^ 01101001     ---------
----------     ----------     ----------      01010000
  01110101       00101010       11000011
```

More Operations on Bits

- Bit-shift operators: << left shift, >> right shift

```
01010101 << 2   is 01010100
                   2 high-order bits shifted out
                   2 low-order bits filled with 0
01101010 << 4   is 10100000
01010101 >> 2   is 00010101
01101010 >> 4   is 00000110

10101100 >> 2   is 00101011 (logical shift)
                or 11101011 (arithmetic shift)
```

Arithmetic right shift: fills high-order bits w/sign bit

C automatically decides which to use based on type:

signed: arithmetic, unsigned: logical

Try out some 4-bit examples:

bit-wise operations:

- 0101 & 1101
- 0101 | 1101

Logical (unsigned) bit shift:

- 1010 << 2
- 1010 >> 2

Arithmetic (signed) bit shift:

- 1010 << 2
- 1010 >> 2

Try out some 4-bit examples:

bit-wise operations:

- 0101 & 1101 = 0101
- 0101 | 1101 = 1101

Logical (unsigned) bit shift:

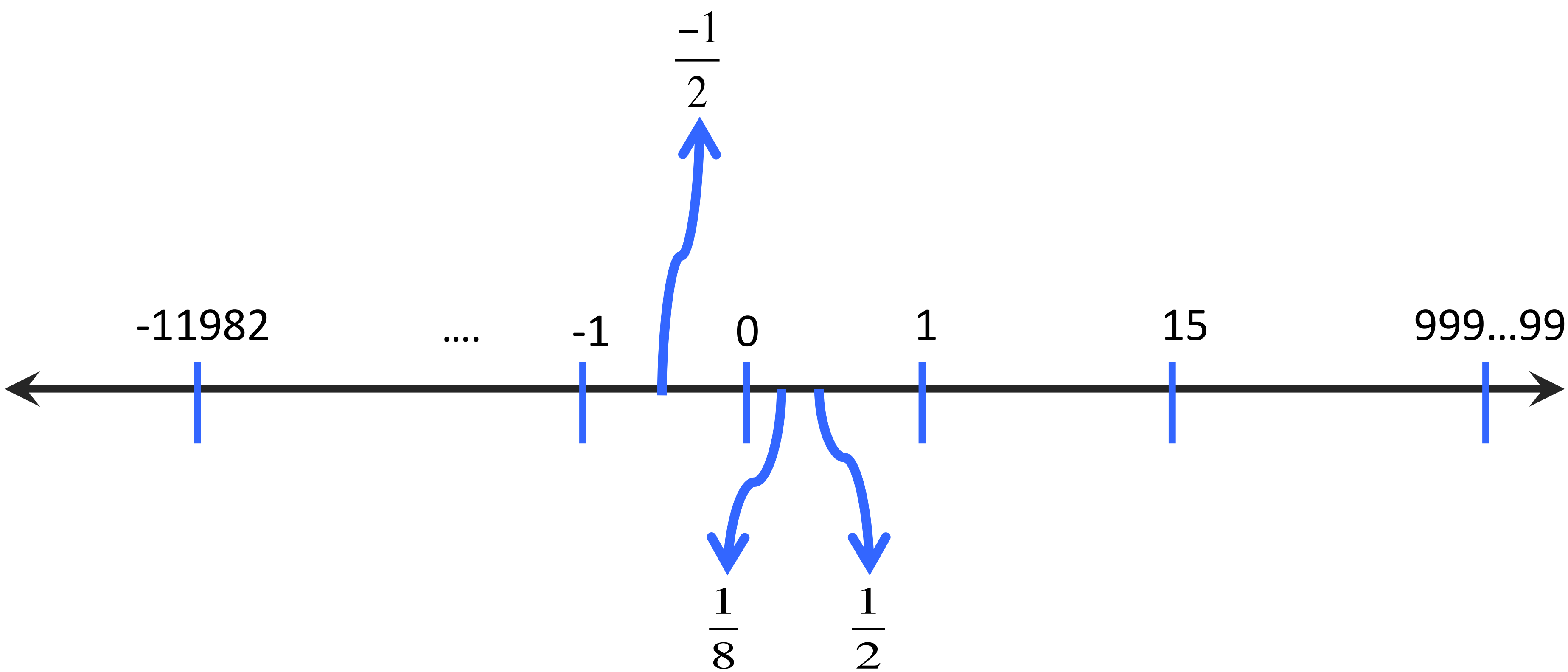
- 1010 << 2 = 1000
- 1010 >> 2 = 0010

Arithmetic (signed) bit shift:

- 1010 << 2 = 1000
- 1010 >> 2 = 1110

Additional Info: Fractional binary numbers

How do we represent fractions in binary?



Additional Info: Representing Signed Float Values

- One option (used for floats, NOT integers)
  - Let the first bit represent the sign
  - 0 means positive
  - 1 means negative

- For example:
  - 0101 -> 5
  - 1101 -> -5

- Problem with this scheme?

Additional Info: Floating Point Representation

1 bit for sign
8 bits for exponent
23 bits for precision

sign | exponent | fraction |

$$\text{value} = (-1)^{\text{sign}} * 1.\text{fraction} * 2^{(\text{exponent}-127)}$$

let's just plug in some values and try it out

```
0x40ac49ba: 0 10000001   01011000100100110111010
      sign = 0 exp = 129   fraction = 2902458
```

$= 1*1.2902458*2^{2} = 5.16098$

I don't expect you to memorize this

Up Next

- C programming