Reading Quiz
Abstraction

User / Programmer
Wants low complexity

Applications
Specific functionality

Software library
Reusable functionality

Operating system
Manage resources

Complex devices
Compute & I/O
Last Class

Data representation

• Binary
• Hexadecimal
• Converting between bases
Today

- Signed-representation
- Binary Arithmetic
- Integer overflow
- Bitwise operations
Unsigned Integers

• Suppose we had one byte
  – Can represent $2^8$ (256) values
  – If unsigned (strictly non-negative): 0 – 255

 252 = 11111100
253 = 11111101
254 = 11111110
255 = 11111111

What if we add one more?

Car odometer “rolls over”.

Slide 6
Unsigned Integers

• Suppose we had one byte
  – Can represent $2^8$ (256) values
  – If unsigned (strictly non-negative): 0 – 255

252 = 11111100
253 = 11111101
254 = 11111110
255 = 11111111

What if we add one more?

Modular arithmetic: Here, all values are modulo 256.
Unsigned Addition (4-bit)

• Addition works like grade school addition:

\[
\begin{array}{c}
1 \\
0110 \quad 6 \\
+ 0100 \quad + 4 \\
1010 \quad 10 \\
\end{array}
\]

Four bits give us range: 0 - 15
Unsigned Addition (4-bit)

- Addition works like grade school addition:

```
    1
  0110   6
+ 0100   + 4
  1010   10
```

Overflow!
Suppose we want to support signed values too (positive and negative). Where should we put -1 and -127 on the circle? Why?

A

-127 (11111111)

0

-1

B

-1 (11111111)

0

-127

C: Put them somewhere else.
Signed Magnitude

• One bit (usually left-most) signals:
  – 0 for positive
  – 1 for negative

For one byte:

\[ 1 = 00000001, \quad -1 = 10000001 \]

Pros: Negation is very simple!
Signed Magnitude

• One bit (usually left-most) signals:
  – 0 for positive
  – 1 for negative

For one byte:

0 = 00000000

What about 10000000?

Major con: Two ways to represent zero.
Two’s Complement (signed)

• Borrow nice property from number line:

```
-1  0  1
```

Only one instance of zero!
Implies: -1 and 1 on either side of it.
Two’s Complement

• Borrow nice property from number line:

Only one instance of zero!
Implies: -1 and 1 on either side of it.
Two’s Complement

• Only one value for zero
• With N bits, can represent the range:
  \(-2^{N-1}\) to \(2^{N-1} - 1\)
• First bit still designates positive (0) / negative (1)

• Negating a value is slightly more complicated:
  \(1 = 00000001\), \(-1 = 11111111\)

From now on, unless we explicitly say otherwise, we’ll assume all integers are stored using two’s complement! This is the standard!
Two’s Compliment

- Each two’s compliment number is now:
\[-2^{n-1}d_{n-1} + 2^{n-2}d_{n-2} + \ldots + 2^{1}d_{1} + 2^{0}d_{0}\]

Note the negative sign on just the first digit. This is why first digit tells us negative vs. positive.
If we interpret 11001 as a two’s complement number, what is the value in decimal?

- Each two’s complement number is now:
  \[ -2^{n-1}d_{n-1} + 2^{n-2}d_{n-2} + \ldots + 2^1d_1 + 2^0d_0 \]

A. -2
B. -7
C. -9
D. -25
“If we interpret...”

• What is the decimal value of 1100?

• ...as unsigned, 4-bit value: 12 (%u)
• ...as signed (two’s comp), 4-bit value: -4 (%d)

• ...as an 8-bit value: 12
  (i.e., \texttt{00001100})
Two’s Complement Negation

• To negate a value $x$, we want to find $y$ such that $x + y = 0$.

• For $N$ bits, $y = 2^N - x$
Negation Example (8 bits)

- For N bits, \( y = 2^N - x \)
- Negate 00000010 (2)
  - \( 2^8 - 2 = 256 - 2 = 254 \)
- Our wheel only goes to 127!
  - Put -2 where 254 would be if wheel was unsigned.
  - 254 in binary is 11111110

Given 11111110, it’s 254 if interpreted as unsigned and -2 interpreted as signed.
Negation Shortcut

• A much easier, faster way to negate:
  – Flip the bits (0’s become 1’s, 1’s become 0’s)
  – Add 1

• Negate 00101110 (46)
  – \(2^8 - 46 = 256 - 46 = 210\)
  – 210 in binary is 11010010
Addition & Subtraction

• Addition is the same as for unsigned
  – One exception: different rules for overflow
  – Can use the same hardware for both

• Subtraction is the same operation as addition
  – Just need to negate the second operand...

• $6 - 7 = 6 + (-7) = 6 + (\sim 7 + 1)$
  – $\sim 7$ is shorthand for “flip the bits of 7”
Subtraction Hardware

Negate and add 1 to second operand:

Can use the same circuit for add and subtract:

\[ 6 - 7 = 6 + \sim7 + 1 \]

input 1 --------------------------->
input 2 --> possible bit flipper --> ADD CIRCUIT ---> result
possible +1 input------->
By switching to two’s complement, have we solved this value “rolling over” (overflow) problem?

A. Yes, it’s gone.

B. Nope, it’s still there.

C. It’s even worse now.

This is an issue we need to be aware of when adding and subtracting!
Overflow, Revisited

Unsigned

Signed

Danger Zone

0 1

-127 127

-128

192 64

128
If we add a positive number and a negative number, will we have overflow? (Assume they are the same # of bits)

A. Always
B. Sometimes
C. Never
Signed Overflow

– Overflow: IFF the sign bits of operands are the same, but the sign bit of result is different.
  • Not enough bits to store result!

Signed addition (and subtraction):

\[
\begin{align*}
2+(-1) &= 1 & 2+(-2) &= 0 & 2+(-4) &= -2 \\
0010 &+ 1111 &= 10001 & 0010 &+ 1110 &= 1110 & 0010 &+ 1100 &= 1110
\end{align*}
\]

No chance of overflow here - signs of operands are different!
Signed Overflow

- Overflow: IFF the sign bits of operands are the same, but the sign bit of result is different.
  - Not enough bits to store result!

Signed addition (and subtraction):

\[
\begin{align*}
2+(-1) &= 1 & 2+(-2) &= 0 & 2+(-4) &= -2 & 2+7 &= -7 & -2+(-7) &= 7 \\
0010 & & 0010 & & 0010 & & 0010 & & 1110 \\
+1111 & & +1110 & & +1100 & & +0111 & & +1001 \\
1 & 0001 & 1 & 0000 & 1110 & 1 & 0001 & 1 & 0111
\end{align*}
\]

Overflow here! Operand signs are the same, and they don’t match output sign!
Overflow Rules

• Signed:
  – The sign bits of operands are the same, but the sign bit of result is different.

• Can we formalize unsigned overflow?
  – Need to include subtraction too, skipped it before.
Recall Subtraction Hardware

Negate and add 1 to second operand:
Can use the same circuit for add and subtract:

\[ 6 - 7 = 6 + \sim 7 + 1 \]

input 1 \( \longrightarrow \) input 2 --\( \longrightarrow \) possible bit flipper --\( \longrightarrow \) possible +1 input --\( \longrightarrow \) ADD CIRCUIT \( \longrightarrow \) result

Let’s call this +1 input: “Carry in”
How many of these unsigned operations have overflowed?

4 bit unsigned values (range 0 to 15):

<table>
<thead>
<tr>
<th>carry-in</th>
<th>carry-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition (carry-in = 0)</td>
<td></td>
</tr>
<tr>
<td>9 + 11 = 1001 + 1011 + 0 = 1 0100</td>
<td></td>
</tr>
<tr>
<td>9 + 6 = 1001 + 0110 + 0 = 0 1111</td>
<td></td>
</tr>
<tr>
<td>3 + 6 = 0011 + 0110 + 0 = 0 1001</td>
<td></td>
</tr>
</tbody>
</table>

| Subtraction (carry-in = 1) |
| 6 - 3 = 0110 + 1100 + 1 = 1 0011 |
| 3 - 6 = 0011 + 1010 + 1 = 0 1101 |

A. 1  
B. 2  
C. 3  
D. 4  
E. 5
How many of these unsigned operations have overflowed?

Interpret these as 4-bit unsigned values (range 0 to 15):

<table>
<thead>
<tr>
<th>Operation</th>
<th>carry-in</th>
<th>carry-out</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Addition (carry-in = 0)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 + 11 = 1001 + 1011 + 0 = 1 0100 = 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 + 6 = 1001 + 0110 + 0 = 0 1111 = 15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 + 6 = 0011 + 0110 + 0 = 0 1001 = 9</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Subtraction (carry-in = 1)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 - 3 = 0110 + 1100 + 1 = 1 0011 = 3</td>
<td>(-3)</td>
<td></td>
</tr>
<tr>
<td>3 - 6 = 0011 + 1010 + 1 = 0 1101 = 13</td>
<td>(-6)</td>
<td></td>
</tr>
</tbody>
</table>

A. 1
B. 2
C. 3
D. 4
E. 5

Pattern?
Overflow Rule Summary

• Signed overflow:
  – The sign bits of operands are the same, but the sign bit of result is different.

• Unsigned: overflow
  – The carry-in bit is different from the carry-out.

<table>
<thead>
<tr>
<th>$C_{in}$</th>
<th>$C_{out}$</th>
<th>$C_{in}$ XOR $C_{out}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

So far, all arithmetic on values that were the same size. What if they’re different?
Sign Extension

- When combining signed values of different sizes, expand the smaller to equivalent larger size:

```c
char y=2, x=-13;
short z = 10;

z = z + y;                z = z + x;
```

```
0000000000001010          0000000000000101
+ 000000010               + 11110011
0000000000000010          11111111111110011
```

Fill in **high-order bits** with **sign-bit** value to get same numeric value in larger number of bytes.
Let’s verify that this works

4-bit signed value, sign extend to 8-bits, is it the same value?

0111  --->  0000 0111  obviously still 7
1010  --->  1111 1010  is this still -6?

-128 + 64 + 32 + 16 + 8 + 0 + 2 + 0 = -6  yes!
Operations on Bits

• For these, doesn’t matter how the bits are interpreted (signed vs. unsigned)

• Bit-wise operators (AND, OR, NOT, XOR)

• Bit shifting
Bit-wise Operators

- bit operands, bit result (interpret as you please)

<table>
<thead>
<tr>
<th>&amp; (AND)</th>
<th></th>
<th>(OR)</th>
<th>~(NOT)</th>
<th>^ (XOR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A &amp; B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>0 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0 1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1 0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1 1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

01010101 & 01101010 = 10101010 ~10101111
00100001 | 10111011 = 01101001 01010000
01110101 & 00101010 = 11000011

More Operations on Bits

- Bit-shift operators: $\ll$ left shift, $\gg$ right shift

01010101 $\ll$ 2 is 01010100
  2 high-order bits shifted out
  2 low-order bits filled with 0

01101010 $\ll$ 4 is 10100000

01010101 $\gg$ 2 is 00010101
01101010 $\gg$ 4 is 00000110

10101100 $\gg$ 2 is 00101011 (logical shift)
  or 11101011 (arithmetic shift)

Arithmetic right shift: fills high-order bits w/sign bit
C automatically decides which to use based on type:
  signed: arithmetic, unsigned: logical
Up Next

• C programming