The Complexity of Computing a Nash Equilibrium

Background Presentation
Note: I have changed the reading

- AGT chapter 2 covers very similar material.
- This is one of the most approachable among the many, many research papers on this topic.
- The full proofs are in a previous paper, which I have spared you.
Decision Problems vs. Total Search Problems

- P & NP are generally stated in terms of decision problems.
  - Is the formula satisfiable?
- Nash is a trivial decision problem.
  - return True
- A search problem requires returning a solution if one exists.
  - A satisfying assignment
  - A Nash equilibrium
- A total search problem is one where a solution always exists.
  - Factoring
  - Nash
Nash \in \text{NP} \ (\text{search version})

- Candidate solution: \( \sigma \), a mixed strategy profile.
- We can verify that \( \sigma \) is a Nash equilibrium in polynomial time.

```python
for each player \( i \):
    for each of player \( i \)'s strategies \( s_i \):
        if \( \text{EU}_i(s_i, \sigma_{-i}) > \text{EU}_i(\sigma) \):
            return False
    return True
return True
```
Fixed Point Theorems

● Fixed points
  ○ \( F(x) = x \)
  ○ The function doesn't change the value at its fixed points.

● Brouwer fixed point theorem
  ○ A continuous function \( f \) mapping a compact convex set into itself always has a fixed point.
  ○ **compact**: closed and bounded
  ○ **convex**: all line segments stay in the set
  ○ **continuous**: \( |f(x+\varepsilon) - f(x)| < \delta \)

en.wikipedia.org/wiki/Fixed-point_theorem#List_of_fixed_point_theorems
Nash’s Theorem

- The best-response correspondence has a fixed-point.

- \( \text{BR}(\sigma) = \{\sigma' \mid U_i(\sigma'_i, \sigma_{-i}) \geq U_i(s_i, \sigma_{-i}) \ \forall \ i, s_i\} \)

- Nash’s proof gives a function \( f \), where \( f(\sigma) \in \text{BR}(\sigma) \), and \( f \) satisfies the conditions of the Brouwer fixed point theorem.
Graphical Game

- Players are graph nodes.
- Utility depends only on your strategy and those of your neighbors.
- Often gives a more-compact representation than the full payoff matrix.