If \((x, p)\) is DSIC, what does \(p\) have to look like?

Recall \(x(b) : \mathbb{R}_+^n \rightarrow X\)

\[ p(b) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \]

If we fix \(i\) and hold \(b_{-i}\) constant, we can think of about how the allocation + payment functions vary with \(b_i\):

\(x_i(b_i)\) what \(i\) gets allocated

\(p_i(b_i)\) what \(i\) pays

\[ \begin{array}{c|c|c}
\hline
b_i & \mathbb{V}ickrey & \mathbb{G}SP \\
\hline
x_i(b_i) & \hline & \\
\hline
p_i(b_i) & \hline & \\
\hline
\end{array} \]
Assume \( \exists p \) s.t. \((x, p) \) is DSIC, and let \( 0 \leq z < y \).

Consider cases

1) \( V_i = z \quad b_i = y \)
   \( \text{lie + bid above value} \)

2) \( V_i = y \quad b_i = z \)
   \( \text{lie + bid below value} \)

By DSIC \( U_i(\text{True bid}) \geq U_c(\text{False bid}) \)

Case 1:
\[
2 X_i(z) - P_i(z) \geq 2 X_i(y) - P_i(y)
\]

Case 2:
\[ y x_i(y) - p_i(y) \geq y x_i(z) - p_i(z) \]
Rearrange both cases to bound \( p_i(y) - p_i(z) \)

1. \( p_i(y) - p_i(z) \geq z (x_i(y) - x_i(z)) \)
2. \( p_i(y) - p_i(z) \leq y (x_i(y) - x_i(z)) \)

Claim: \( x \) implementable \( \implies \) \( x \) monotone.
Proof (by contradiction):
Suppose \( x \) is not
Monotone. Then there exists some $i$, $b_i$, $0 \leq z < y$ s.t. $X_i(y) < X_i(z)$, so $|X_i(y) - X_i(z)| < 0$.

By 1,2 we know

$y < z < y$, but

since $< 0$ and $y > z$,

we know $z > y$,

a contradiction.

We therefore conclude that our assumption ($x$ not monotone) cannot hold.
Claim: \( x \) monotone and piecewise constant \( \implies x \) implementable.

proof (by construction) we will build a payment rule \( p \) s.t. \((x, p)\) is DSIC.

Fix \( z \) and let \( y \downarrow z \)

case 1: \( \chi_i(z) = \lim_{y \downarrow z} \chi_i(y) \)

by the payment difference sandwich, \( 0 \leq p_i(y) - p_i(z) \leq 0 \), so \( p_i(z) = p_i(y) \).

case 2: \( \chi_i(z) < \lim_{y \downarrow z} \chi_i(y) \)
\[ z_i(y) - x_i(z) \leq p_i(y) - p_i(z) \leq y_i(x_i(y) - x_i(z)) \]

\[ z_i(y) - x_i(z) = p_i(y) - p_i(z) \]

Therefore, if \( p_i(0) = 0 \), the unique DSIC payment rule is:

\[
p_i(b_i) = \sum\limits_{Z_j} Z_j \left( \lim_{z \to Z_j} x_i(z) \right) - \left( \lim_{z \uparrow Z_j} x_i(z) \right)
\]

where \( Z_j \) ranges over the discontinuities in \( x_i \) up to \( b_i \).

Are we done?

We've narrowed down the choices to one candidate.
possibilities is one condition, but we need to show it's DSIC. 

_sub-claim: \( \chi \) monotone \( \Rightarrow \) \((\chi, p^*) \) is DSIC.

proof (by picture)

\[ b_i \quad v_i \quad \]

\[ v_i \quad b_i \quad \]
Myerson’s Lemma:
In a single-parameter environment,
1) $\chi$ implementable $\iff \chi$ monotone

2) $\rho$ is unique:
$$\sum_{\text{jumps}} (\text{location of jump}) (\text{size of jump})$$

for piecewise constant $\chi$. 