Mixed Strategies

4/25/18
Pursuit/Evasion Game
Exercise: Construct a payoff matrix.

Available actions:
- Left, Right

Incentives:
- Catch: +1
- Miss: -1
- +5
- +3
- 0
Exercise: Identify all Nash equilibria.

- There are no pure-strategy equilibria!
- We need mixed strategies
  - Agents should deliberately randomize their actions so that the opponent can’t predict them.
Mixed Strategies

Players can choose a probability distribution over their actions.

For example, could go left with probability 0.4, and right with probability 0.6.

Mixed strategy:
\[ \langle 0.4, 0.6 \rangle \]

How should respond against \( \langle 0.4, 0.6 \rangle \)?
Responding to Mixed Strategies

The *best responses* to a mixed strategy are the pure strategies with the highest expected value.

Consider the strategy \( \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \rangle \) in Rock-Paper-Scissors.

- \( U_1(R, \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \rangle) \) denotes P1’s expected value for playing \( R \) against P2’s mixed strategy \( \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \rangle \).
Expected Value in Mixed Strategies

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\[
U_1\left(R, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \right\rangle \right) = \frac{1}{2}U_1(R, R) + \frac{1}{4}U_1(R, P) + \frac{1}{4}U_1(R, S)
\]

\[
= \frac{1}{2}(0) + \frac{1}{4}(-1) + \frac{1}{4}(1) = 0
\]

\[
U_1\left(P, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \right\rangle \right) = \frac{1}{2}(1) + \frac{1}{4}(0) + \frac{1}{4}(-1) = \frac{1}{4}
\]

\[
U_1\left(S, \left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \right\rangle \right) = \frac{1}{2}(-1) + \frac{1}{4}(1) + \frac{1}{4}(0) = -\frac{1}{4}
\]

Paper is the best response.
Mixed-Strategy Nash Equilibrium

A Nash equilibrium is a mixed strategy for each player, where every player’s strategy is a best response to the others’ strategies.

How can a mixed strategy be a best response?

- Only possible if all of the actions with non-zero probability are best responses.
Rock-Paper-Scissors Nash Equilibrium

First verify that there are no dominated strategies and no pure-strategy equilibria.

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\[
U_1 \left( R, \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle \right) = \frac{1}{3}(0) + \frac{1}{3}(-1) + \frac{1}{3}(1) = 0
\]

\[
U_1 \left( P, \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle \right) = \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}(-1) = 0
\]

\[
U_1 \left( S, \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle \right) = \frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(0) = 0
\]

R, P, and S are all best responses to \( \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle \) for P1.
Rock-Paper-Scissors Nash Equilibrium

By essentially the same calculations, R, P, and S are all best responses to \( \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle \) for P2.

\[
U_2 \left( \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle, R \right) = \frac{1}{3}(0) + \frac{1}{3}(-1) + \frac{1}{3}(1) = 0
\]

\[
U_2 \left( \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle, P \right) = \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}(-1) = 0
\]

\[
U_2 \left( \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle, S \right) = \frac{1}{3}(-1) + \frac{1}{3}(1) + \frac{1}{3}(0) = 0
\]

Therefore, both players playing mixed strategy \( \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle \) is a Nash equilibrium.
A Tougher Example

Suppose winning with R rocks!

Should you play R more often, less often, or equally often than \( \frac{1}{3} \)?

Key insight: solve for the probabilities that make the other player(s) indifferent.

\[
Pr(R) = \frac{4}{12} \\
Pr(P) = \frac{5}{12} \\
Pr(S) = \frac{3}{12}
\]
Exercise: Find the Mixed-Strategy NE

Step 1: find the probabilities can play to make indifferent between L and R.

Step 2: find the probabilities can play to make indifferent between L and R.

\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & 0,1 & 5,-1 \\
R & 3,-1 & 0,1 \\
\end{array}
\]
Mixed-Strategy Support

The *support* of a mixed strategy is the set of actions that are played with non-zero probability.

In all of the examples so far, all players have used full-support mixed strategies in equilibrium.

Once we know the right support for every player, finding the probabilities requires solving a system of linear equations (linear programming).

Finding the right supports is actually the hard part.
General Algorithm for Nash Equilibria

eliminate dominated strategies
search for pure strategy equilibria

for each possible combination of supports:
  NE = find equilibrium with given supports
for each player:
  BR = best response to NE
  if BR ∉ player’s support:
    NE is not an equilibrium

There are exponentially many supports, so this algorithm takes exponential time.
• It is an open problem whether a sub-exponential algorithm exists.
More than two agents

- Add more dimensions to the array.
- Add more utilities to each outcome tuple.

In a game with $N$ agents that each have $S$ actions, how many utilities does the payoff matrix contain?