Normal Form Games

2-12-16
Game Representations

Extensive Form Game
- Agents take turns
- Represented as a tree.
- Internal nodes are agent decisions.
- Edges are actions.
- Leaf nodes are outcomes.

Normal Form Game
- Agents move simultaneously.
- Represented as a matrix.
- Each agent controls a dimension.
- Actions select a row/column.
- Matrix cells are outcomes.

R
0,0 -1,1 1,-1
P 1,-1 0,0 -1,1
S

1

2

L R

L R L R

(3,1) (1,2) (2,1) (0,0)
Components of normal form games

<table>
<thead>
<tr>
<th>players (agents)</th>
<th>strategies (actions, policies)</th>
<th>more players add more dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>all players have a utility for each outcome</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0,0</td>
<td>-1,1</td>
<td>3,-1</td>
</tr>
<tr>
<td>B</td>
<td>1,-1</td>
<td>0,5</td>
<td>1,1</td>
</tr>
<tr>
<td>C</td>
<td>-2,2</td>
<td>1,-2</td>
<td>0,-3</td>
</tr>
</tbody>
</table>

payoffs (utilities)
How should agents pick the best action?

First idea: dominance

Strategy T dominates strategy B, because it is a better response to every action player 2 could take.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>5,2</td>
<td>2,3</td>
<td>3,4</td>
</tr>
<tr>
<td>M</td>
<td>4,1</td>
<td>3,2</td>
<td>4,0</td>
</tr>
<tr>
<td>B</td>
<td>3,3</td>
<td>1,2</td>
<td>2,2</td>
</tr>
</tbody>
</table>

Knowing B will never be chosen, we can eliminate it from the game, and iterate.
Exercise: iteratively eliminate dominated strategies

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>5,5</td>
<td>2,8</td>
</tr>
<tr>
<td>B</td>
<td>8,2</td>
<td>-6,-6</td>
</tr>
<tr>
<td>C</td>
<td>4,10</td>
<td>-9,-8</td>
</tr>
<tr>
<td>D</td>
<td>-6,3</td>
<td>-8,1</td>
</tr>
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Hint: this game is symmetric, so if a strategy is dominated for player 1, it is also dominated for player 2 (check this).
What if dominance doesn’t eliminate everything?

Key idea: Nash equilibrium

An equilibrium is an outcome where no player can gain by unilaterally deviating.

- A profile where agents’ strategies are mutual best responses.
Exercise: identify all (pure-strategy) equilibria

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<th>C</th>
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<tbody>
<tr>
<td>E</td>
<td>-7,-7</td>
<td>-4,-7</td>
<td>3,-2</td>
<td>-10,5</td>
</tr>
<tr>
<td>F</td>
<td>-7,-2</td>
<td>2,-1</td>
<td>-9,-8</td>
<td>-4,-4</td>
</tr>
<tr>
<td>G</td>
<td>-2,3</td>
<td>-3,-9</td>
<td>2,0</td>
<td>0,1</td>
</tr>
<tr>
<td>H</td>
<td>3,-9</td>
<td>0,-9</td>
<td>-1,-1</td>
<td>0,9</td>
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What if there’s no pure-strategy equilibrium?

Key idea: mixed strategies

- players can randomize over the available actions
Mixed strategy Nash equilibrium

No player can gain (in expectation) by switching to any pure strategy.

\[ \text{EV}( R, [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] ) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 + \frac{1}{3} \cdot 1 = 0 \]

\[ \text{EV}( P, [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] ) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 = 0 \]

\[ \text{EV}( S, [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] ) = \frac{1}{3} \cdot -1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0 \]

Thus both players using \([\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]\) is a Nash equilibrium.
Clicker Question: what equilibrium do you expect?

If winning with rock is worth twice as much, what changes?

a) higher probability of rock
b) higher probability of paper
c) higher probability of scissors
d) same equilibrium

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Mixed-strategy NE requires indifference

Key idea: solve for the probability that makes the other players indifferent among the strategies you want them to mix.

\[
\begin{align*}
EV(R) &= p_R \cdot 0 + p_P \cdot -1 + p_S \cdot 2 \\
EV(P) &= p_R \cdot 1 + p_P \cdot 0 + p_S \cdot -1 \\
EV(S) &= p_R \cdot -1 + p_P \cdot 1 + p_S \cdot 0 \\
EV(R) &= EV(P) = EV(S)
\end{align*}
\]

4 equations, 4 unknowns

\[
p_R + p_P + p_S = 1
\]

we can solve for \(p_R, p_P, p_S\)

\[
\begin{bmatrix}
\frac{4}{12} \\
\frac{5}{12} \\
\frac{3}{12}
\end{bmatrix}
\]
Exercise: confirm the equilibrium

\[
\text{EV}(R, \left[ \frac{4}{12}, \frac{5}{12}, \frac{3}{12} \right]) = ?
\]

\[
\text{EV}(P, \left[ \frac{4}{12}, \frac{5}{12}, \frac{3}{12} \right]) = ?
\]

\[
\text{EV}(S, \left[ \frac{4}{12}, \frac{5}{12}, \frac{3}{12} \right]) = ?
\]