Lower Bounds for Gap-Hamming-Distance and Consequences for Data Stream Algorithms

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Counting Distinct Elements in a Data Stream

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Input: Stream of integers $\sigma =< a_1, \ldots, a_m >$

Output: F_0 := number of distinct elements in σ

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Goal: Minimize space used to compute F_0

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Frequency Moments: $F_k = \sum_{i=1}^{n} \mathrm{freq}(i)^k$ [Alon-Matias-Szegedy '96]

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Do multiple passes help? If not, why not?

The Gap-Hamming-Distance Problem

Input: Alice gets $x \in \{0,1\}^n$, Bob gets $y \in \{0,1\}^n$.

Output:

- $\text{GHD}(x,y) = 1$ if $\Delta(x,y) > \frac{n}{2} + \sqrt{n}$
- $\text{GHD}(x,y) = 0$ if $\Delta(x,y) < \frac{n}{2} - \sqrt{n}$

Problem: Design randomized, constant error protocol to solve this

Cost: Worst case number of bits communicated

$x =$	**0**	**1**	**0**	**0**	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**
$y =$	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**0**	**1**

$$n = 12; \quad \Delta(x,y) = 3 \in [6 - \sqrt{12}, 6 + \sqrt{12}]$$

The Reductions

E.g., Distinct Elements (Other problems: similar)

$x =$	**0**	**1**	**0**	**0**	**1**	**0**	**1**	**1**	**0**	**0**	**0**	**1**
σ :	(1,0)	(2,1)	(3,0)	(4,0)	(5,1)	(6,0)	(9,0)	(8,0)	(9,0)	(10,0)	(11,0)	(12,1)
$y =$	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**0**	**1**
τ :	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)	(6,0)	(9,0)	(8,0)	(9,1)	(10,0)	(11,0)	(12,1)

Alice: $x \longmapsto \sigma = \langle (1, x_1), (2, x_2), \ldots, (n, x_n) \rangle$

Bob: $y \longmapsto \tau = \langle (1, y_1), (2, y_2), \ldots, (n, y_n) \rangle$

Notice: $F_0(\sigma \circ \tau) = n + \Delta(x, y) = \begin{cases} < \frac{3n}{2} - \sqrt{n}, & \text{or} \\ > \frac{3n}{2} + \sqrt{n}. & \end{cases}$ Set $\varepsilon = \frac{1}{\sqrt{n}}$.

Communication to Streaming

p-pass streaming algorithm $\Longrightarrow$ $(2p-1)$-round communication protocol

messages $=$ memory contents of streaming algorithm

And Thus

Previous results [Indyk-Woodruff'03], [Woodruff'04], [C.-Cormode-McGregor'07]:

- For one-round protocols, $\mathrm{R}^{\rightarrow}(\textsc{ghd}) = \Omega(n)$
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Key open questions:

- What is the unrestricted randomized complexity $\mathrm{R}(\textsc{ghd})$?
- Better algorithm for Distinct Elements (or F_k, or H) using two passes?

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- Simplification, clever reduction from $\textsc{index}$ [Jayram-Kumar-Sivakumar]

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What we show:

- Theorem 1: $\Omega(n)$ lower bound for any $O(1)$-round protocol

  Holds under uniform distribution
- Theorem 2: one-round, deterministic: $\mathrm{D}^{\to}(\textsc{ghd}) = n - \Theta(\sqrt{n}\log n)$
- Theorem 3: $\mathrm{R}^{\to}(\textsc{ghd}) = \Omega(n)$ (simpler proof, uniform distrib)

  (independently proved by [Woodruff'09])

Technique: Round Elimination

Base Case Lemma: There is no "nice" 0-round GHD protocol.

Round Elimination Lemma: If there is a "nice" k-round GHD protocol, then there is a "nice" $(k-1)$-round GHD protocol.

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- The $(k-1)$-round protocol will be solving a "simpler" problem
- Parameters degrade with each round elimination step

Parametrized Gap-Hamming-Distance Problem

The problem:

$$\mathrm{GHD}_{c,n}(x,y) = \begin{cases} 1\,, & \text{if } \Delta(x,y) \geq n/2 + c\sqrt{n}\,, \\ 0\,, & \text{if } \Delta(x,y) \leq n/2 - c\sqrt{n}\,, \\ \star\,, & \text{otherwise.} \end{cases}$$

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Protocol assumptions (eventually, will lead to contradiction):

- Deterministic k-round protocol for $\mathrm{GHD}_{c,n}$
- Each message is $s \ll n$ bits
- Error probability $\leq \varepsilon$, under distribution $\mu_{c,n}$

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Main Construction: Given k-round protocol $\mathcal{P}$ for $\text{GHD}_{c,n}$, construct $(k-1)$-round protocol $\mathcal{Q}$ for $\text{GHD}_{c',n'}$

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- Protocol $\mathcal{Q}_1$:
  - Input: $x', y' \in \{0,1\}^A$ where $A \subseteq [n]$, $|A| = n'$
  - Extend $x' \to x$ s.t. Alice sends m on input x
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  - BAD_1: $\text{GHD}_{c',n'}(x',y') \neq \text{GHD}_{c,n}(x,y)$.
  - BAD_2: $\text{GHD}_{c,n}(x,y) \neq \mathcal{P}(x,y)$.

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VC-Dimension

Fixing Alice's first message:

- Call x good if $\Pr_y[\mathcal{P}(x, y) \neq \text{GHD}_{c,n}(x, y)] \leq 2\varepsilon$
  Then $\#\{\text{good } x\} \geq 2^{n-1}$ (Markov)
- Let $M = M_{\mathsf{m}} = \{\text{good } x : \text{Alice sends m on input } x\}$.
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Shattering:

- Say $S \subseteq \{0,1\}^n$ shatters $A \subseteq [n]$ if $\#\{x|_A : x \in S\} = 2^{|A|}$
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Extend $x' \to x$: pick $x \in M$ such that $x' = x|_A$

The First Bad Event

Recall BAD_1: $\textsc{ghd}_{c',n'}(x', y') \neq \textsc{ghd}_{c,n}(x, y)$.

Notation: $x = x' \circ \bar{x}$, $y = y' \circ \bar{y}$, $n = n' + \bar{n}$.

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Lemma: If $\bar{x}, \bar{y}$ nearly orthogonal and $c' \geq 2c$, then

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Corollary: $\Pr[BAD_1] < 1/8$.

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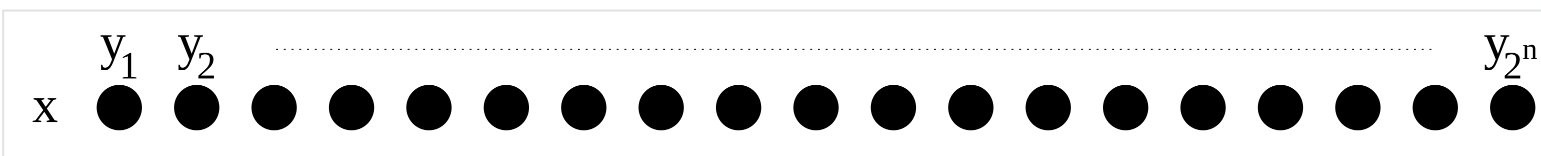
Recall BAD_2: $\textsc{ghd}_{c,n}(x, y) \neq \mathcal{P}(x, y)$.

Bounding $\Pr[BAD_2]$ is subtle:

- x is good, so $\Pr[\mathcal{P}\text{ errs} \mid x] \leq 2\varepsilon$
  - But this requires $(x, y) \sim \mu_{c,n}$
- Random extension $(x', y') \to (x, y)$ is not $\sim \mu_{c,n}$.



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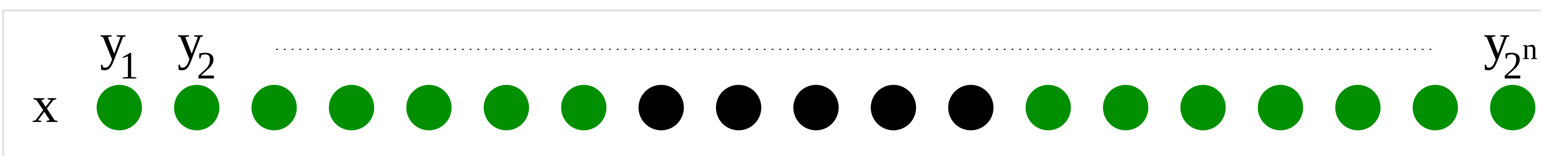

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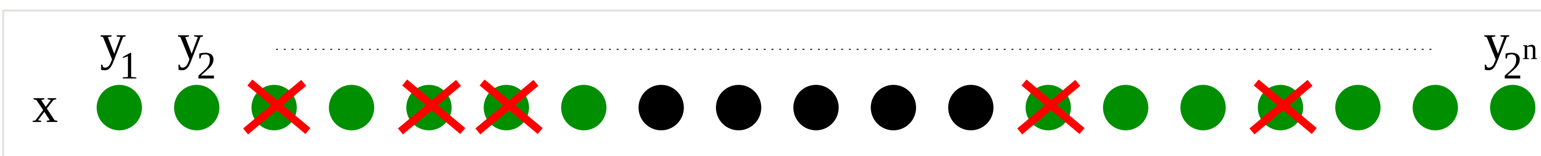

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- Actual distrib (fixed x, random y):
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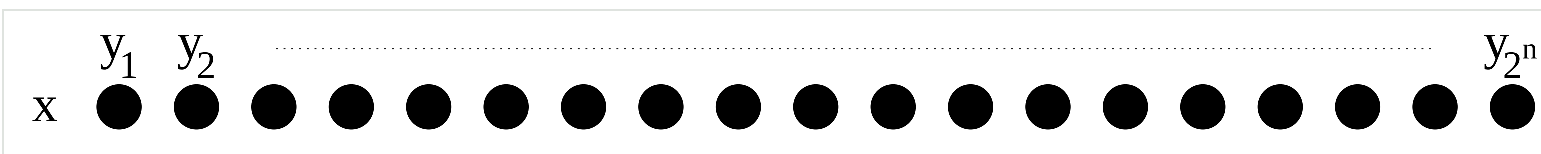
  - y uniform over a subset of $\{0, 1\}^n$, just like in $\mu_{c,n}$



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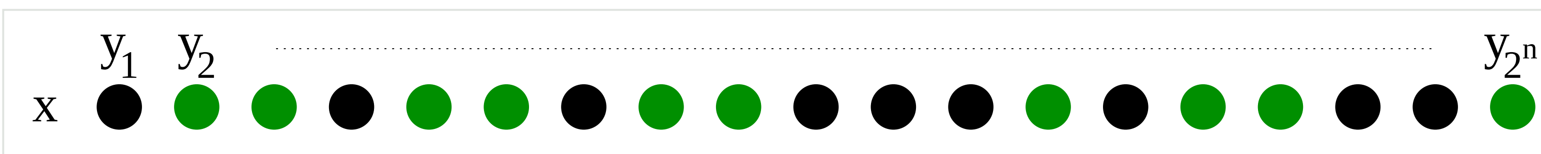

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Recall BAD_2: $\textsc{GHD}_{c,n}(x,y) \neq \mathcal{P}(x,y)$.

Bounding $\Pr[BAD_2]$ is subtle:

- x is good, so $\Pr[\mathcal{P}$ errs $\mid x] \leq 2\varepsilon$
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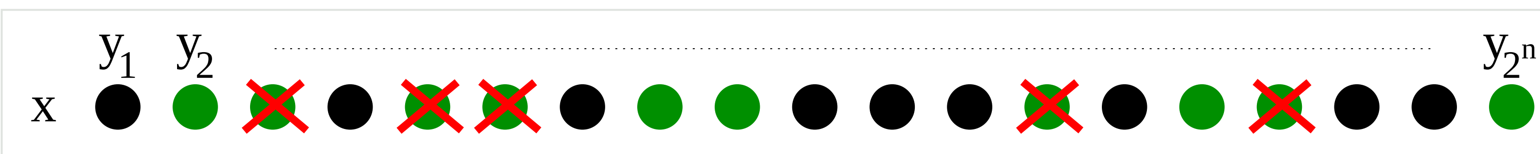

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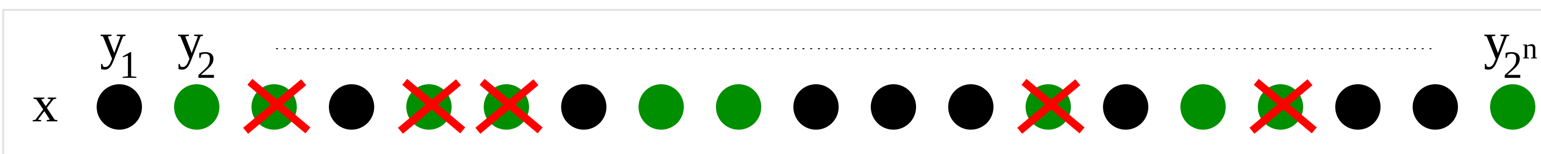

Lemma: $\Pr[BAD_2] = O(\varepsilon)$.

Round Elimination, First Attempt (Recap)

Putting it together:

- $\mathcal{P}$ is k-round ε-error protocol for $\text{GHD}_{c,n}$
- $\mathcal{Q}_1$ is $(k-1)$-round ε'-error protocol for $\text{GHD}_{c',n'}$ with
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Lemma: If there is a k-round, ε-error protocol for $\text{GHD}_{c,n}$ in which each player sends $s \ll n$ bits, then there is a $(k-1)$-round, $O(\varepsilon)$-error protocol for $\text{GHD}_{2c,n/3}$ in which each player sends $2^{O(k)}s$ bits.

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More Specific: $\mathrm{R}^{k}(\mathrm{GHD}) = n/2^{O(k^2)}$.

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  We're biased (Amit helped invent it, so it's his pet technique)

Open Problems

1. The key problem here: Settle $\mathrm{R}(\textsc{ghd})$.
2. More generally: Understand communication complexity of "gap problems" better.
3. This should help with other streaming problems, e.g., longest increasing subsequence.

Questions? Comments? Post-Doc/Job offers?

Contact jbrody@cs.dartmouth.edu