The Probabilistic Method

Week 8: Second Moment Method

Joshua Brody
CS49/Math59
Fall 2015
What is a graph property?

(A) a set of graphs
(B) a set of graphs closed under addition of edges
(C) a set of graphs closed under addition of vertices
(D) a set of graphs closed under isomorphism
(E) None of the above
What is a graph property?

(A) a set of graphs
(B) a set of graphs closed under addition of edges
(C) a set of graphs closed under addition of vertices
(D) a set of graphs closed under isomorphism
(E) None of the above
The First Moment Method

Basic Method:

(1) Define bad events $\text{BAD}_i$
(2) $\text{BAD} := \bigcup_i \text{BAD}_i$
(3) bound $\Pr[\text{BAD}_i] \leq \delta$
(4) Compute # bad events $\leq m$
(5) union bound:
   $\Pr[\text{BAD}] \leq m\delta < 1$
(6) $\therefore \Pr[\text{GOOD}] > 0$

as First Moment Method:

(1) $Z_i$: indicator var for $\text{BAD}_i$
(2) $Z := \sum_i Z_i$
(3) $E[Z_i] = \Pr[\text{BAD}_i] \leq \delta$
(4) Compute # bad events $\leq m$
(5) $E[Z] = E[Z_i] \leq m\delta < 1$
(6) $\therefore Z = 0 \text{ w/prob} > 0$
Exploiting Expected Value

Suppose $X$ is non-negative, integer random variable

Fact: $\Pr[X > 0] \leq E[X]$
Exploiting Expected Value

Suppose $X$ is non-negative, integer random variable

**Fact:** $\Pr[X > 0] \leq E[X]$

**Consequences:**

- If $E[X] < 1$, then $\Pr[X=0] > 0$
- If $E[X] = o(1)$, then $\Pr[X=0] = 1-o(1)$
- If $E[X] \to \infty$, then ???
More on The Second Moment Method

Theorem: \( \Pr[X = 0] \leq \frac{\text{Var}[X]}{\text{E}[X]^2} \)

proof:

- use Chebyshev’s Inequality with \( \alpha := \text{E}[X] \)
More on The Second Moment Method

**Theorem:** \[ \Pr[X = 0] \leq \frac{\text{Var}[X]}{\text{E}[X]^2} \]

**proof:**

- use Chebyshev’s Inequality with \( \alpha := \text{E}[X] \)

**Consequences:**

- If \( \text{Var}[X] = o(\text{E}[X]^2) \), then \( \Pr[X=0] = o(1) \)
More on The Second Moment Method

Theorem: \( \Pr[X = 0] \leq \frac{\text{Var}[X]}{\text{E}[X]^2} \)

proof:
- use Chebyshev’s Inequality with \( \alpha := \text{E}[X] \)

Consequences:
- If \( \text{Var}[X] = o(\text{E}[X]^2) \), then \( \Pr[X=0] = o(1) \)
More on The Second Moment Method

**Theorem:** \( \Pr[X = 0] \leq \frac{\text{Var}[X]}{E[X]^2} \)

**proof:**
- use Chebyshev’s Inequality with \( \alpha := E[X] \)

**Consequences:**
- If \( \text{Var}[X] = o(E[X]^2) \), then \( \Pr[X=0] = o(1) \)
- If \( \text{Var}[X] = o(E[X]^2) \), then \( X \sim E[X] \) almost always.

\( X > 0 \) “almost always”
Random Graphs

\[ G \sim G(n,p) : \text{random graph on } n \text{ vertices } V = \{1, \ldots, n\} \]

each edge \((i,j) \in E\) independently with prob. \(p\)

\[ G(n,p) : \text{probability distribution} \]

\(G : \text{random variable}\)
When are $A_S$ and $A_T$ not independent?

(A) S, T share at least one vertex
(B) S, T share at least one edge
(C) S, T share at least two vertices
(D) (A) and (B)
(E) (B) and (C)
Clicker Question

When are \( A_S \) and \( A_T \) not independent?

(A) \( S, T \) share at least one vertex
(B) \( S, T \) share at least one edge
(C) \( S, T \) share at least two vertices
(D) (A) and (B)
(E) (B) and (C)
The Probabilistic Method

some of us see the world in terms of expected value, we are very different from the rest of you.

www.chalkboardmanifesto.com