The Probabilistic Method

Week 8: Second Moment Method

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CS49/Math59
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Reading Quiz

What is a graph property?

(A) a set of graphs

(B) a set of graphs closed under addition of edges

(C) a set of graphs closed under addition of vertices

(D) a set of graphs closed under isomorphism

(E) None of the above

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The First Moment Method

Basic Method:

(1) Define bad events **BAD_i**

(2) **$BAD := \cup_i \ BAD_i$**

(3) bound **$Pr[BADi] \leq \delta$**

(4) Compute # bad events **$\leq m$**

(5) union bound:

$Pr[BAD] \leq m\delta < 1$

(6) $\therefore$ **$Pr[GOOD] > 0$**

as First Moment Method:

(1) **Z_i**: indicator var for **BAD_i**

(2) **$Z := \sum_i Z_i$**

(3) **$E[Z_i] = Pr[BAD_i] \leq \delta$**

(4) Compute # bad events **$\leq m$**

(5) **$E[Z] = E[Z_i] \ \leq m\delta < 1$**

(6) $\therefore$ **$Z = 0$ w/prob > 0**

Exploiting Expected Value

Suppose **X** is non-negative, integer random variable

Fact: **Pr[X > 0] ≤ E[X]**

Exploiting Expected Value

Suppose X is non-negative, integer random variable

Fact: $\Pr[X > 0] \leq E[X]$

Consequences:

- If $E[X] < 1$, then $\Pr[X=0] > 0$
- If $E[X] = o(1)$, then $\Pr[X=0] = 1-o(1)$
- If $E[X] \to \infty$, then ???

More on The Second Moment Method

Theorem: **$\Pr[X = 0] \leq \mathrm{Var}[X]/E[X]^2$**

proof:

- use Chebyshev's Inequality with α **:= E[X]**

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Consequences:

- If $\mathrm{Var}[X] = o(E[X]^2)$, then $\Pr[X=0] = o(1)$

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$X > 0$ "*almost always*"

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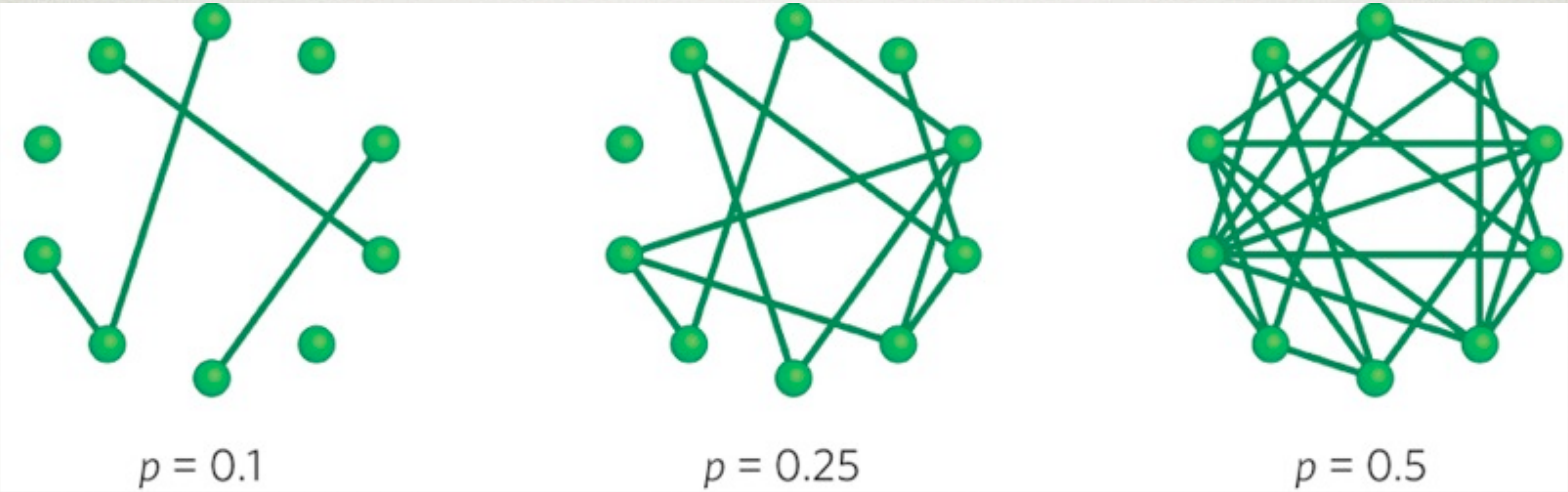
- If $\mathrm{Var}[X] = o(E[X]^2)$, then $\Pr[X=0] = o(1)$
- If $\mathrm{Var}[X] = o(E[X]^2)$, then $X \sim E[X]$ almost always.

Random Graphs

[Erdős-Rényi 60]

G ~ G(n,p) : random graph on **n** vertices **V = {1, ..., n}**
each edge **(i,j) ∈ E** independently with prob. **p**



G(n,p) : probability distribution
G : random variable

Clicker Question

When are A_S and A_T ***not*** independent?

(A) S, T share at least one vertex

(B) S, T share at least one edge

(C) S, T share at least two vertices

(D) (A) and (B)

(E) (B) and (C)

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