The Probabilistic Method

Week 8: Second Moment Method

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CS49/Math59
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Reading Quiz

What is Cov[X,Y]?

(A) **Cov[X,Y] = E[X]E[Y] - E[XY]**

(B) **Cov[X,Y] = E[XY] - E[X]E[Y]**

(C) **Cov[X,Y] = E[(X-E[X])2]**

(D) **Cov[X,Y] = Var[XY] - E[XY]**

(E) **None of the above**

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About how many prime factors does a typical integer **n** have?

(A) $\sqrt{n}$

(B) log(n)

(C) ln(ln n)

(D) log(ln(ln n))

(E) None of the above

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MOMENTS

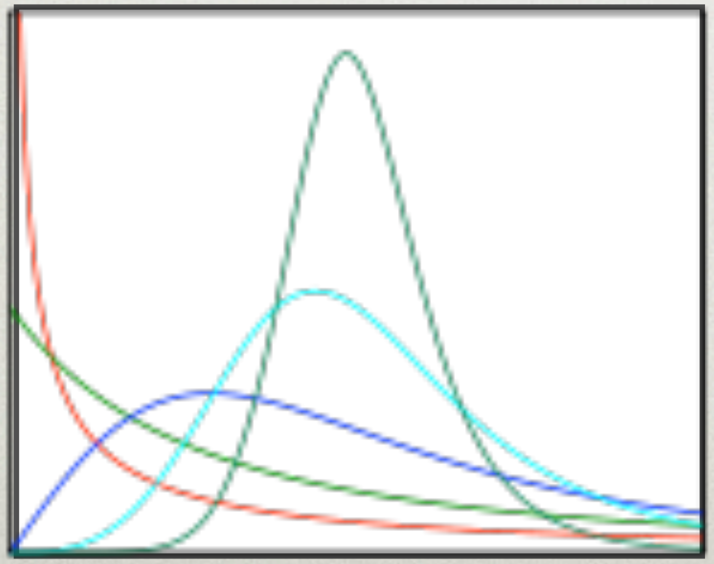
kth *central* moment: **E[(X-E[X])^k]**

(1) expected value: **E[X]**

(2) variance: **E[(X-E[X])²]**

(3) skewness: **E[(X-E[X])³]**

(4) ...



The First Moment Method

Most previous techniques use *First Moment Method*

Basic Method: **as First Moment Method:**

(1) Define bad events $\mathbf{BAD_i}$

(2) $\mathbf{BAD} := \cup_i \; \mathbf{BAD_i}$

(3) bound $\mathbf{Pr[BADi]} \leq \delta$

(4) Compute # bad events $\leq \mathbf{m}$

(5) union bound:

$$\mathbf{Pr[BAD]} \leq \mathbf{m}\delta < 1$$

(6) $\therefore$ $\mathbf{Pr[GOOD] > 0}$

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as First Moment Method:

(1) $\mathbf{Z_i}$: indicator var for $\mathbf{BAD_i}$

(2) $\mathbf{Z := \sum_i Z_i}$

(3) $\mathbf{E[Z_i] = Pr[BAD_i] \leq \delta}$

(4) Compute # bad events $\mathbf{\leq m}$

(5) $\mathbf{E[Z] = E[Z_i] \ \leq m\delta < 1}$

(6) ∴ $\mathbf{Z = 0}$ **w/prob** $\mathbf{> 0}$

Clicker Question

Let **X,Y** be fair coins. What is **Cov[X,Y]** when

(i) **X = Y**

(ii) **X = 1-Y**

(A) **(i) 1/4 (ii) -1/4**

(B) **(i) 1/2 (ii) -1/2**

(C) **(i) -1/4 (ii) 1/4**

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Most Integers Have ln(ln(n)) Prime Factors

Theorem: Let **g(n)** be a function growing arbitrarily slowly. Then, there are **o(n)** integers **x ≤ n** such that

$$|\nu(x) - \ln(\ln(n))| > g(n) \sqrt{\ln(\ln(n))}$$

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Proof:

- Take random $x \in [n]$
- Estimate # distinct primes dividing **x**
- Use Chebyshev to bound deviation.

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