The Probabilistic Method

Week 5: Linearity Of Expectation

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CS49/Math59
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Reading Quiz

X be a real-valued random variable. What is **E[X]**?

(A) $E[X] = \sum_{w \in \Omega} X(w)\Pr[w]$

(B) $E[X] = \sum_{s \in S} s\Pr[X = s]$

(C) $E[X] = (1/\Omega)\sum_{w \in \Omega} \Pr[w]$

(D) **Multiple answers correct**

(E) **None of the above**

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Expected Value

Definition: If $\mathbf{X : \Omega \to S}$ is a real-valued random variable, then its expected value is

$$\mathbf{E[X] = \sum_{w \in \Omega} X(w)\ Pr[w]}$$

Clicker Question

Let **X** be a fair six-sided die. What is **$E[X^2]$**?

(A) **$E[X^2] = 7$**

(B) **$E[X^2] = 28/3$**

(C) **$E[X^2] = 49/4$**

(D) **$E[X^2] = 91/6$**

(E) **None of the above**

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Linearity Of Expectation

Problem: **300** students took an exam for **CS21**. There were **8** problems to solve. Each problem was correctly solved by at least **200** students. Show that there are two students who between them answered all questions correctly.

Sum-Free Sets

A set of integers **A** is **sum-free** if there are no elements $\mathbf{a_1, a_2, a_3} \in \mathbf{A}$ such that $\mathbf{a_1 + a_2 = a_3}$.

sum-free:

{1, 3, 8, 10, 12} **{11, 15, 19, 23, 31}**

not sum-free:

{1, 2, 4, 6, 11} **{0, 14, 139, 150, 200}**

Sum-Free Sets

Theorem: every set of **n** nonzero integers has a sum-free subset of size **> n/3.**

example: integers **mod m**

0	1	2	...	$\frac{m}{3}$	$\frac{m}{3}+1$	...	...	$\frac{2m}{3}$	...	m-2	m-1

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- $a_1 = m/3 + i$
- $a_2 = m/3 + j$

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- $a_1 = m/3 + i$
- $a_2 = m/3 + j$
- $a_1+a_2 > 2m/3, \quad a_1+a_2 \leq m/3$

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