The Probabilistic Method

Week 4: The Basic Method

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CS49/Math59

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images source: wikipedia, google

Clicker Question

What does it mean for a tournament to have property S_k?

(A) **There is a set of k players that beat all other players.**

(B) **For any set of k players, there is one player that beats them all.**

(C) **There is a set of k players that are all beaten by one player.**

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Which of the following sets are sum-free?

(A) **A = {1, 2, 4, 6}**

(B) **B = {17, 19, 35, 47, 101}**

(C) **C = {-14, 22, 57, 71}**

(D) **multiple answers correct**

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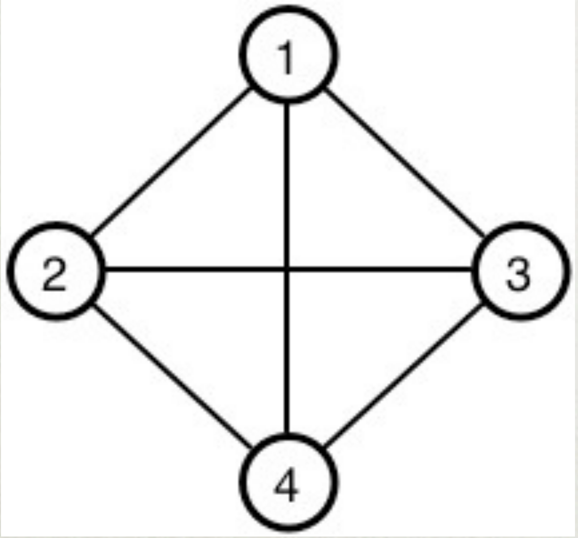
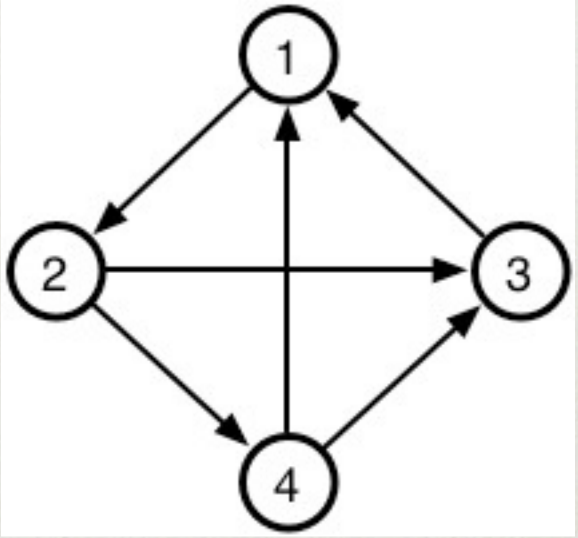
Tournaments

Definition: A tournament on **n** players is an **orientation** of $\mathbf{K_n}$

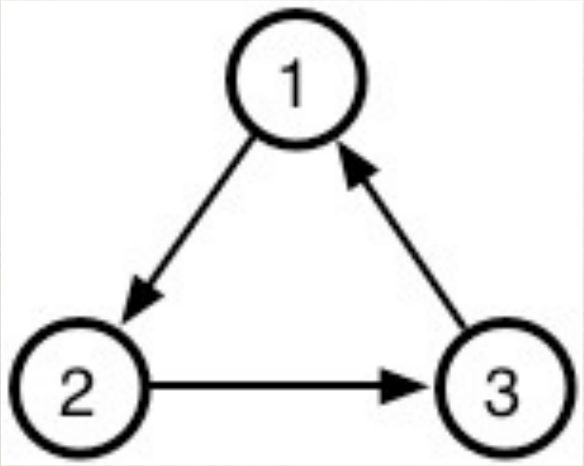


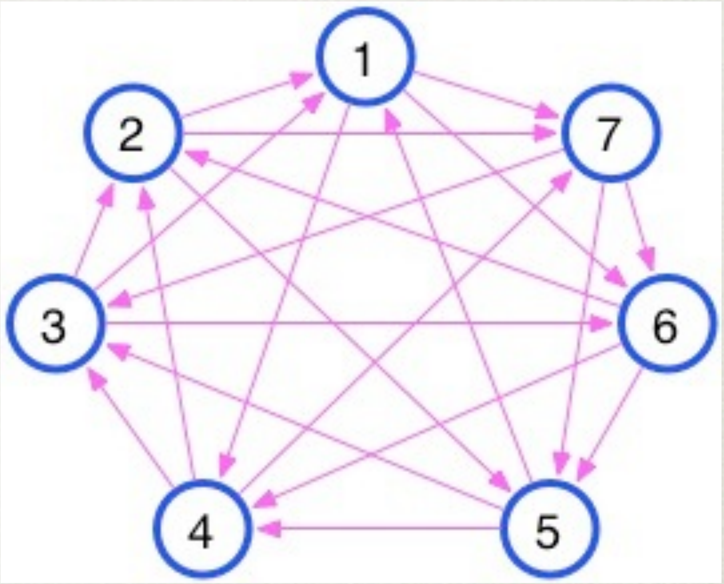


(u,v) directed edge: **"u beats v"**

Property S_k

Definition: T has property S_k if every set of k vertices there is another vertex that beats them all.





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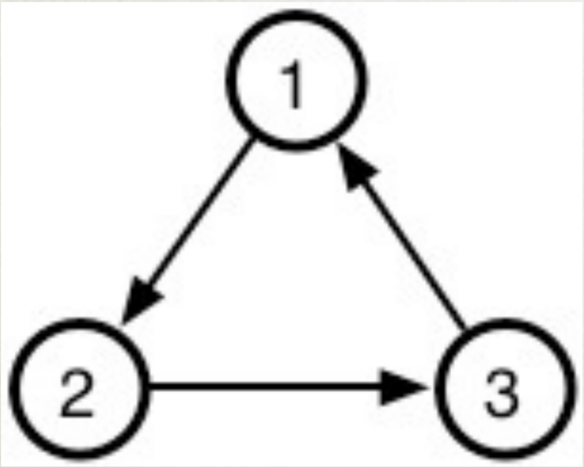

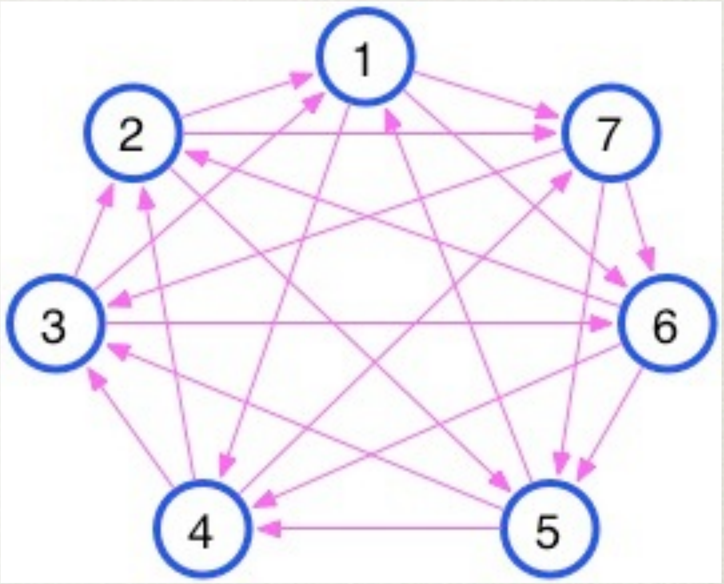

Question: Are there always tournaments w/ S_k?

Tournaments with S_k

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union bound:

$$\Pr[BAD] \leq \binom{n}{k}(1-2^{-k})^{n-k} < 1$$

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