The Probabilistic Method

Week 4: The Basic Method

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CS49/Math59
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images source: wikipedia, google

Asymptotics Recap

Exponent Rule: If $g(n) = f(n) + \omega(1)$, then

$$2^{f(n)} = o(2^{g(n)})$$

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Comparing Functions with Different Bases:

(1) Convert to common base using e.g. $a = 2^{\log(a)}$

(2) Use Exponent Rule

Clicker Question

What is **K_n**?

(A) **A complete graph on n vertices**

(B) **An independent set on n vertices**

(C) **A complete graph with n edges**

(D) **A clique with n edges**

(E) **Multiple answers correct**

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What is **R(k,l)**?

(A) **The least n such that it is possible to color edges of K_n red or blue so that there is a red clique w/k vertices and a blue clique with ℓ vertices.**

(B) **The least n such that no matter how edges of K_n are two-colored, there is a red clique w/k-vertices and a blue clique w/ℓ vertices.**

(C) **The least n such that no matter how edges of K_n are two-colored, there is a red clique w/k-vertices or a blue clique w/ℓ vertices.**

(D) **The greatest n such that there is a two-coloring of K_n with no k-vertex red clique and no ℓ-vertex blue clique.**

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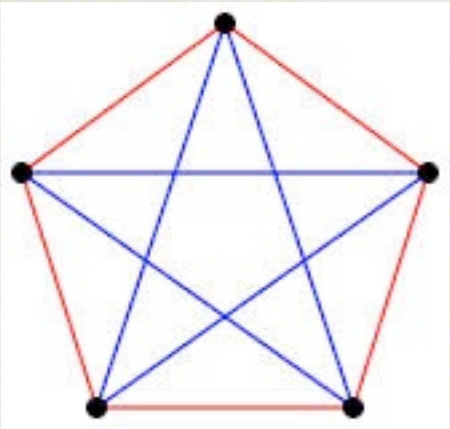
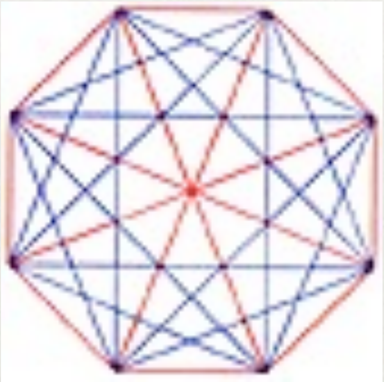
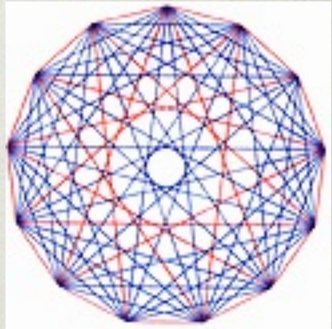
Ramsey Theory

Definition: **R(k,l) =** smallest **n** such that for every two-coloring of **K_n**, there is red **K_k** subgraph or a blue **K_ℓ** subgraph.







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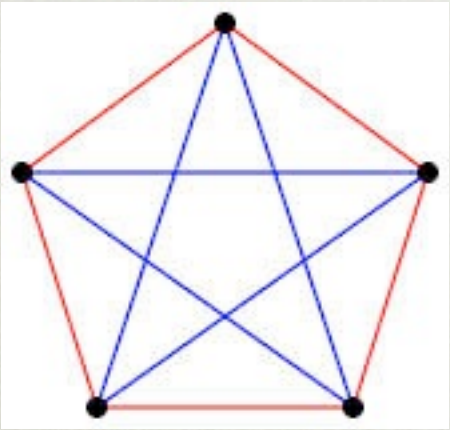
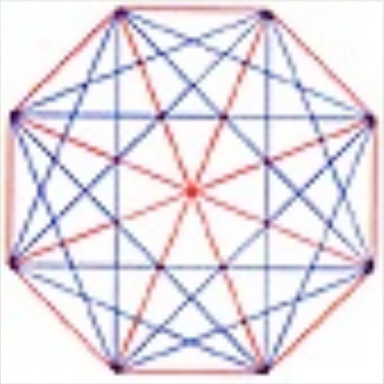
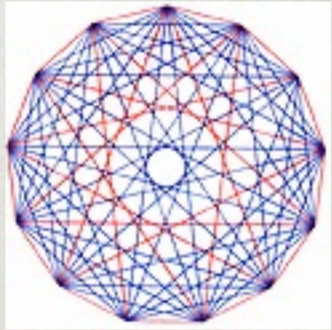


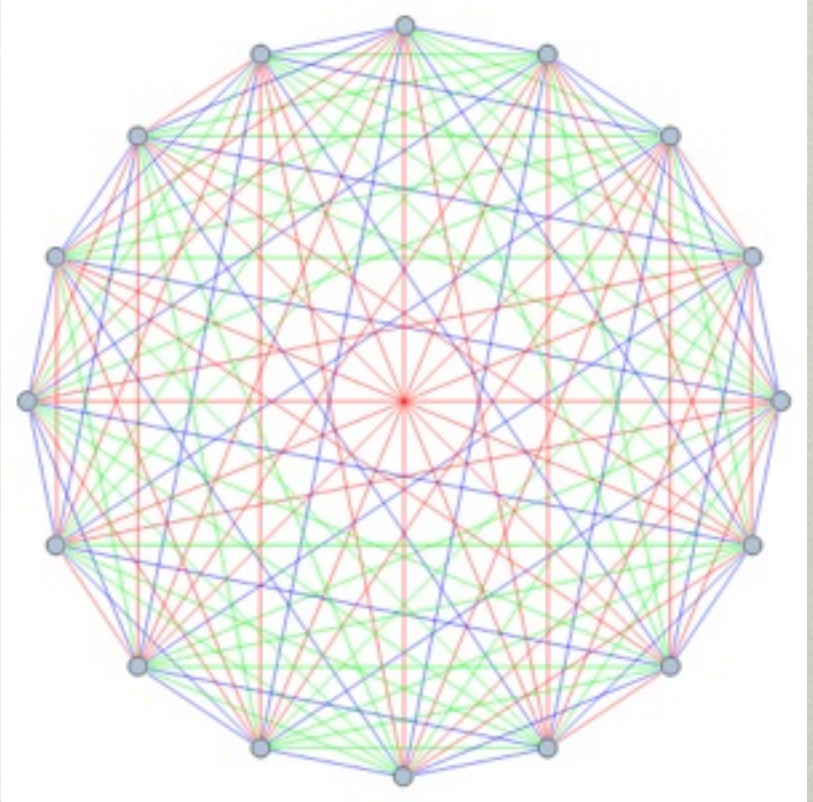




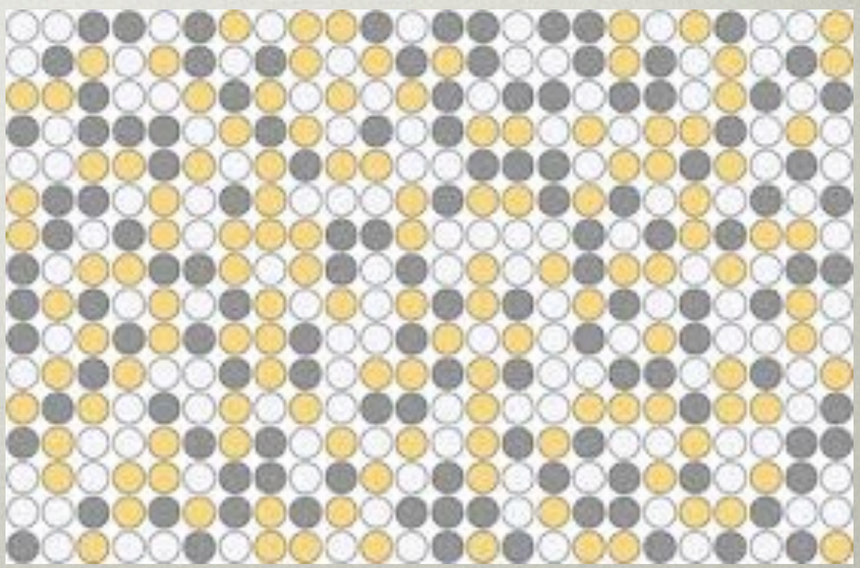
Ramsey's Theorem [Ramsey '29]

R(k,l) is finite for all **k,l**.

Ramsey Theory Extensions



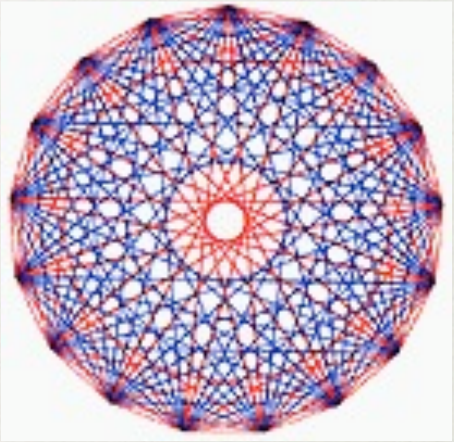
- **$R(k_1,k_2,...,k_t)$**: smallest **n** so that in any **t**-coloring of edges of **K_n**, either there is a red clique of size **k_1**, a blue clique of size **k_2**, a purple clique of size **k_3**, ...
- **$R(G_1,G_2,...,G_t)$**: smallest **n** so that any **t**-coloring of edges of **K_n** contains an all-red copy of **G_1** or all-blue copy of **G_2**, or...
- Integers: for all **$k \geq 2$**, there is **$n>3$** s.t. in any **k**-coloring of **{1, ..., n}**, there are **x,y,z** all same color where **$x+y=z$**.
- also: hypergraphs, Van der Waerden numbers, ...



Applications

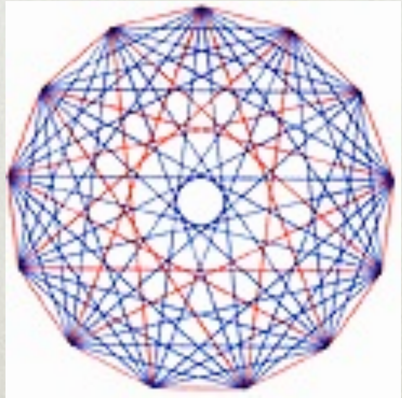
Theoretical Computer Science:

- communication complexity
- information theory
- computational complexity



Algorithms:

- parallel computation
- testing graph properties
- matrix multiplication
- computational geometry
- data structures



Mathematics:

- number theory
- algebra
- topology
- logic
- geometry

R(k,k) Lower Bound

Theorem: If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$.

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- BAD_S: event that edges between vertices in S are monochromatic
- $BAD := \cup_S \; BAD_S$; $GOOD := \neg BAD$

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- # k vertex subsets S: $\binom{n}{k}$

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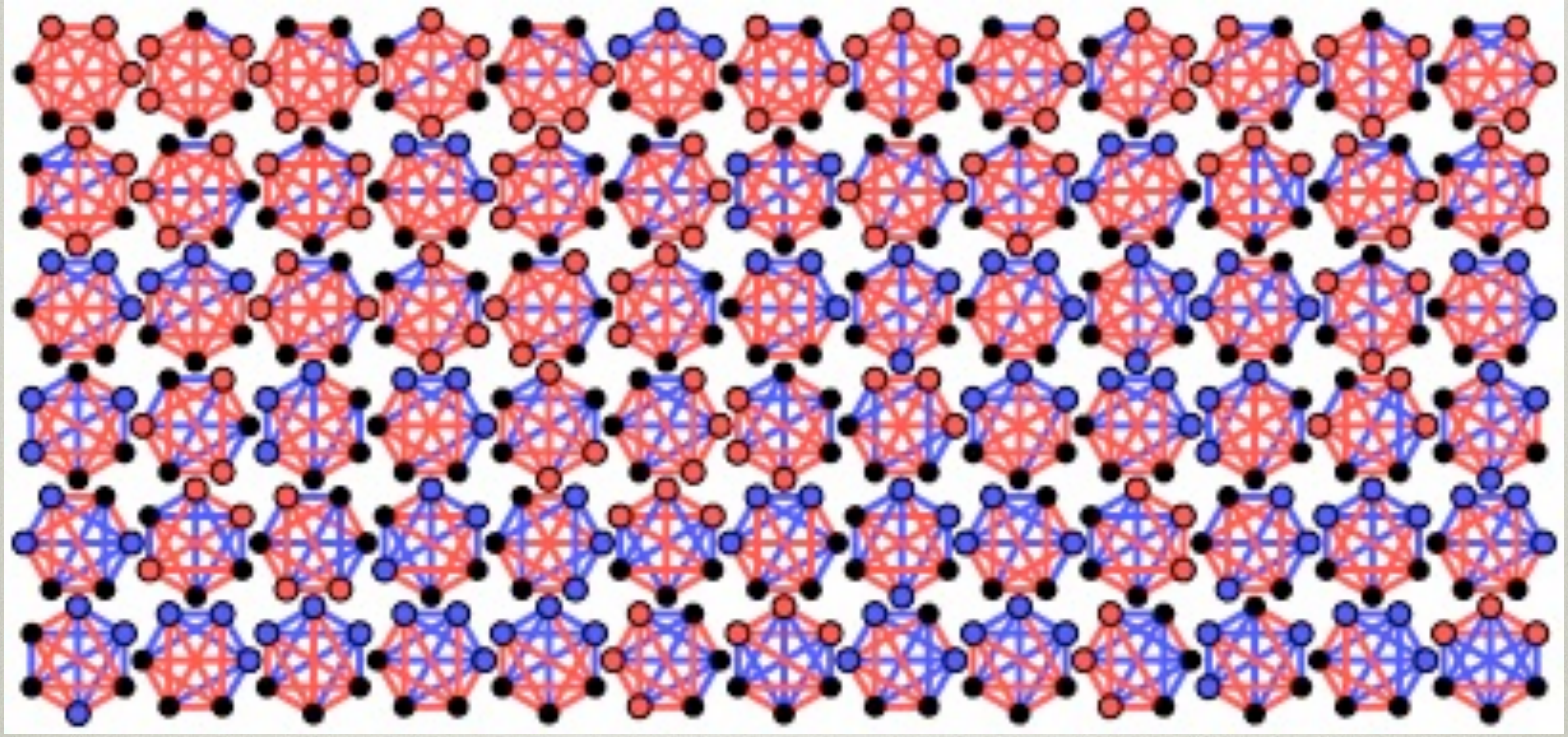
- BAD_S: event that edges between vertices in S are monochromatic
- $BAD := \cup_S \ BAD_S$; $GOOD := \neg BAD$
- $\Pr[BAD_S] = 2^{1-\binom{k}{2}}$
- # k vertex subsets S: $\binom{n}{k}$
- Union Bound: $\Pr[BAD] < \binom{n}{k}2^{1-\binom{k}{2}} < 1$
- $\Pr[GOOD] = 1 - \Pr[BAD] > 0$

Current Bounds For R(k,l)

k,l	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10
3	3	6	9	14	18	23	28	36	40-42
4	4	9	18	25	36-41	49-61	58-84	73-115	92-149
5	5	14	25	43-49	58-87	80-143	101-216	126-316	144-442
6	6	18	36-41	58-87	102-165	113-298	132-495	169-780	179-1171
7	7	23	49-61	80-143	113-298	205-540	217-1031	241-1713	289-2826
8	8	28	58-84	101-216	132-495	217-1031	282-1870	317-3583	331-6090
9	9	36	73-115	126-316	169-780	241-1713	317-3583	565-6588	581-12677
10	10	40-42	92-149	144-442	179-1171	289-2826	331-6090	581-12677	798-23556

source: [Radziszowski14], *wikipedia*



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