The Probabilistic Method

Week 13: P, NP, SAT

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CS49/Math59
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Which of the following languages are known to be in $P$?

(A) THREE-COLOR = \{G = (V,E): G can be colored using three colors so that no edge is monochromatic\}

(B) BIPARTITE = \{G = (V,E): G is bipartite\}

(C) PRIMES = \{integers n: n is a prime number\}

(D) FACTORING = \{(n,k) : n has factor d s.t. 1 < d < k\}

(E) Multiple Answers Correct
QUIZ

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**Clicker Question**

Give a representation of two arbitrarily large integers \((n,k)\) as a single bit string.
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How many bits are in the representation? (ignore ceiling function)

(A) \(\log(n) + \log(k)\)
(B) \(\max[ \log(n), \log(k) ]\)
(C) \(3\log(nk)\)
(D) \(2\max[ \log(n), \log(k)]\)
(E) None of the above
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(C) \(3\log(nk)\)

(D) \(2\max[\ \log(n), \log(k)\]\) \(\text{(other answers possible)}\)

(E) None of the above
**Polynomial Time Verifiers**

V is an *efficient verifier* for a decision problem L if:

1. V is a polynomial time algorithm that takes two inputs: x and w
2. x ∈ L iff there is w such that \( \text{length}(w) = \text{length}(x)^{O(1)} \) and \( V(x, w) = \text{YES} \)
**Polynomial Time Verifiers**

**V** is an efficient verifier for a decision problem **L** if:

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**NP**: = set of languages verifiable in polynomial time
Hard Problems?

Problems for which no polytime algorithm is known:

• **INDEPENDENT-SET**: Given $G = (V,E)$ and integer $k$, is there an *independent set* of size at least $k$?

• **VERTEX-COVER**: Given $G = (V,E)$ and integer $k$, is there a *vertex cover* of size at most $k$?

• **FACTORING**: Given integers $(n,k)$, does $n$ have a factor $1 < d < k$?

• **SUBSET-SUM**: Given set of integers $A = \{a_1, a_2, ..., a_n\}$ is there a *subset* $S \subseteq A$ that *sums to zero*?
Clicker Question

Consider the following SAT instance: \( c_1 \land c_2 \land c_3 \land c_4 \)
for the following clauses:

\[
\begin{align*}
  c_1 &= (x_1 \lor x_2 \lor \neg x_3) \\
  c_2 &= (\neg x_1 \lor x_3 \lor x_4) \\
  c_3 &= (\neg x_2 \lor \neg x_3 \lor \neg x_4) \\
  c_4 &= (x_2 \lor \neg x_3 \lor x_4)
\end{align*}
\]

Which of the following are satisfying assignments?

(A) \((x_1, x_2, x_3, x_4) = (F, F, F, F)\)

(B) \((T, T, T, T)\)

(C) \((T, F, T, F)\)

(D) \((F, T, F, F)\)

(E) Multiple Answers Correct
Clicker Question

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for the following clauses:

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c_1 = (x_1 \lor x_2 \lor \neg x_3) \quad c_2 = (\neg x_1 \lor x_3 \lor x_4) \\

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\quad c_3 = (\neg x_2 \lor \neg x_3 \lor \neg x_4) \quad c_4 = (x_2 \lor \neg x_3 \lor x_4)
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Which of the following are satisfying assignments?

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(B) \( (T, T, T, T) \)

(C) \( (T, F, T, F) \)

(D) \( (F, T, F, F) \)

(E) Multiple Answers Correct
Clicker Question

What is the input length of SAT?
(how would you efficiently encode a SAT input?)

(A) $O(n+m)$
(B) $O(nm)$
(C) $O(n + m\log(n))$
(D) $O(2^n)$
(E) None of the above
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Design an algorithm recognizing SAT.
What is its runtime?

(A) $O(2^n m)$
(B) $O(n^m)$
(C) $O(nm)$
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some of us see the world in terms of expected value. We are very different from the rest of you.

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