The Probabilistic Method

Week 11: Applications, Randomized Algorithms

Joshua Brody

CS49/Math59

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Reading Quiz

Which Frequency Moment can be approximated by a randomized algorithm that uses logarithmic space?

(A) F_0

(B) F_2

(C) F_6

(D) F_{11}

(E) Multiple Answers Correct

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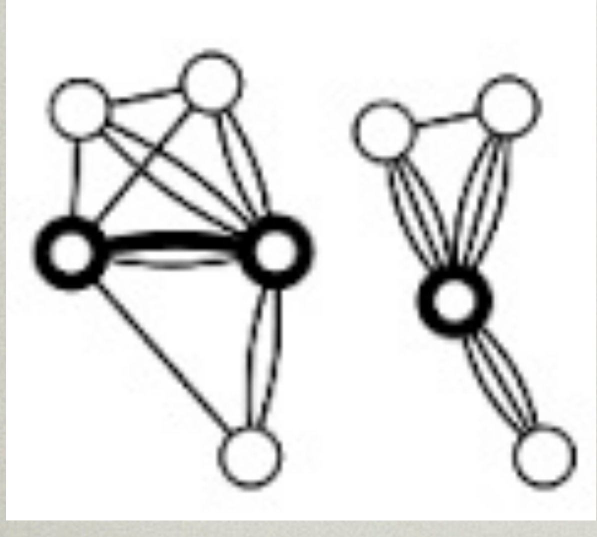
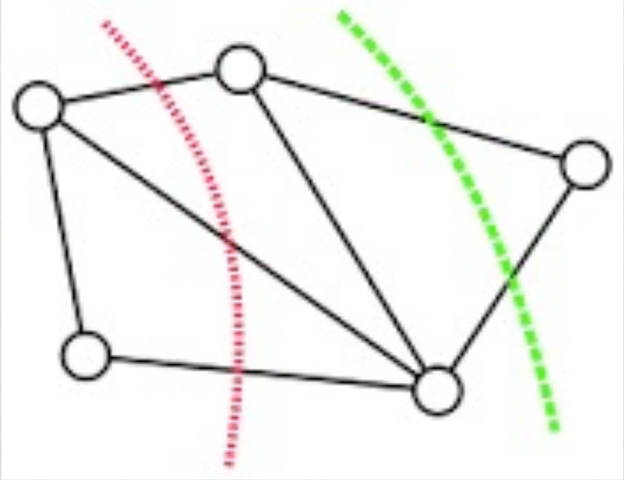
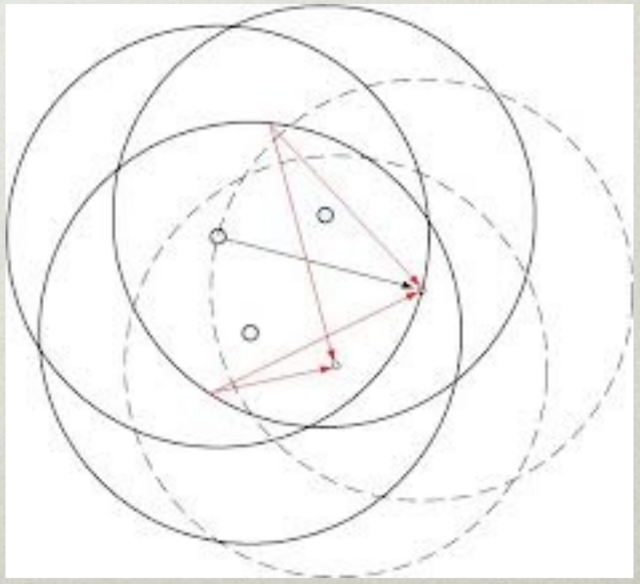
Randomized Algorithms

Types of randomization:

- algorithm uses randomness
  - Las Vegas: *no error, expected runtime*
  - Monte Carlo: *small error, worst-case runtime*
- input is randomized (algorithm often deterministic)

Randomized Algorithms

Why randomized algorithms?

- reduce runtime (or space or ...)
- simpler algorithm
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History

- developed late 1970's
- *computational geometry*
- *primality testing* (deterministic algorithm not known until 2003)
- many applications (machine learning, streaming, ...) assume randomness ***by default***

Clicker Question

You see a series of integers $a_1, a_2, \ldots, a_m \in \{1, \ldots, n\}$ that appear one at a time.

Design a randomized algorithm that outputs a_K for a uniform $1 \leq K \leq m$, using as little space as possible.

How much space is used?

(A) $O(m)$

(B) $O(\sqrt{m})$

(C) $O(\log(m))$

(D) $O(\log(\log(m))$

(E) **None of the above**

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Data Stream Model

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- *Stream:* m elements from universe of size n
  - *e.g.*, *6,9,4,2,1,3,8,8,6,5,7,2,9,3,3,4,7,12, ...*
- *Goal:* Compute function *F* of stream e.g., # *distinct elements, frequency moments, entropy*, ...
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Alon, Matias, Szegedy '96 take home messages:
(i) Randomization usually needed
(ii) Approximation usually needed

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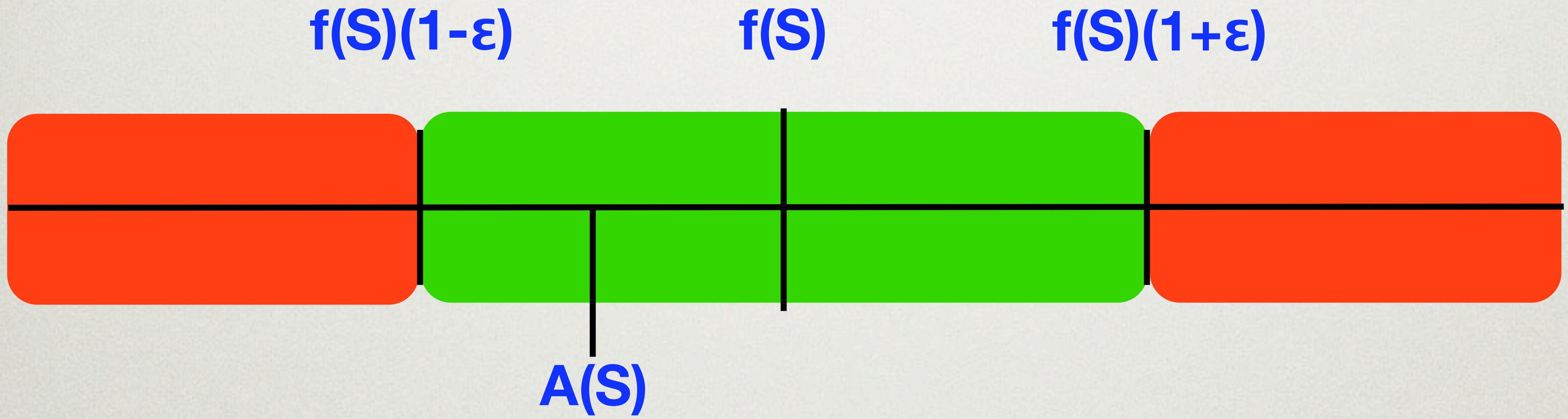
Theorem: [Alon-Matias-Szegedy '96]

Approximating $F_2(s)$ possible in $O(\log(mn)/\varepsilon^2)$ space.

PAC Algorithms

Probably Approximately Correct (PAC) model:

"most of the time we're close enough"



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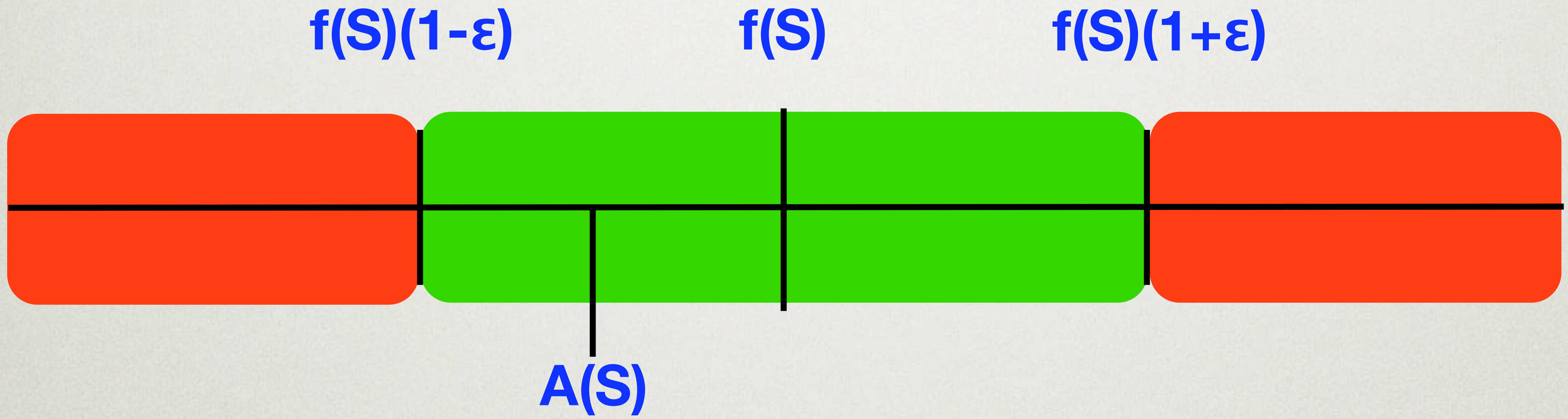

Definition: A streaming algorithm **A** is an (ε,δ)-approximation for **f** if for any stream **S**

$$\Pr[f(s)(1-\varepsilon) \leq A(S) \leq f(s)(1+\varepsilon)] \geq 1-\delta$$

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