CS49/Math59 Lab 6

This lab assignment is due **before the start of class** on Wednesday, November 18. Homework handed in during class but after I begin the lecture will be counted as late submissions. Some things to note:

- This is a **two week lab**. Start early.
- I encourage you to write your solution using LaTeX, but you are not required to.
- Aside from your partner (if you have one), you should not discuss problems in detail with anyone. It's OK to discuss approaches at a high level. In fact, I encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.

Make sure your names are on your submission, and show your work to maximize partial credit.

1. **Zero-One Laws.** In class, we saw that the property $\langle$G has a triangle$\rangle$ can be expressed in first-order theory of graphs, and that therfore, it obeys a zero-one law for any constant $0 < p < 1$. Express the following properties in the first-order theory of graphs:
   (a) G is a complete graph.
   (b) G has diameter ≤ 2. (The *diameter* of a graph is the maximum shortest path length between any two vertices in G.) Having diameter at most 2 means that for any two vertices u, v in G, there is a $u \rightsquigarrow v$ path of length at most two.

2. **First Moment Method.** In class, we've seen that if $E[X] = o(1)$, then $X = 0$ almost always. It might seem like the opposite is true when $E[X] = \omega(1)$ i.e., $X > 0$ almost surely when $E[X] \to \infty$. However, this is false in general.

   Give an example random variable which refutes this claim. Specifically, you should define a random variable X such that (i) as $n \to \infty$, $E[X] \to \infty$ and yet (ii) $X = 0$ almost surely.

3. **Clique Number for Random Graphs.** Let $G \sim G(n, 1/2)$, and define $k := 2\log(n)$. Show that almost always $CL(G) < k$. Hint: use the First Moment Method and sufficiently tight approximations for $\binom{n}{k}$.

4. **Threshold Functions.** In class we saw that the property $\langle CL(G) \geq 4\rangle$ has threshold function $r(n) = n^{-2/3}$. The property $\langle$G has a triangle$\rangle$ has a similar threshold function.
   (a) Give a threshold function $r(n)$ for the property $\langle G$ has a triangle$\rangle$.
   (b) Show that if $p << r(n)$, then G almost always has no triangles.
   (c) Show that if $p >> r(n)$, then G almost always has a triangle.

5. **Dominating Sets.** (Alon/Spencer Exercise 10.1) A *dominating set* in a graph G is a subset of vertices $S \subseteq V$ such that every vertex is either in S or adjacent to a vertex in S. (On Exam 2, these were called *neighboring sets.*)

Show that there is a graph on n vertices with minimum degree at least $n/2$ in which every dominating set has size $\Omega(\log n)$.

Hint: You're welcome to show this any way you want, but one promising line of attack is the following:

(a) Briefly (in a sentence or two) argue that if G has a dominating set of size k, then it must have a dominating set of size $k+1$. Conclude that if G has *no* dominating set of size k then it has no dominating set of size $k' < k$.

(b) Choose $p = 3/4$ and $k = \frac{\log(n)}{16}$,[1] and let $G \sim G(n, p)$. Define BAD_1 to be the event that G has a vertex with degree less than $n/2$. Define BAD_2 to be the event that G has a dominating set of size k.

(c) Use Chernoff bounds to show that $\Pr[BAD_1] < 1/3$.

(d) Use the First Moment Method to show that $\Pr[BAD_2] = o(1)$.

(e) Conclude that there must be some G with minimum degree at least $n/2$ and where each dominating set has size $\Omega(\log n)$.

6. **Attribution.** Did you get assistance on any of the problems on this assignment from anyone aside from me and/or your lab partner? For example, did you discuss any problems at a high level with other students? Did you accidentally stumble on solutions while doing a websearch on related material? If so, descibe the nature of the assistance here. (e.g. "We briefly discussed problem 1 with X,Y, and Z" or "We saw a solution on ⟨this website⟩ before finding our own solution") If you (and your partner) worked alone, please say so here.

7. **Lab Questionnaire.** (None of these questions will have an impact on your grade, this is to help provide the feedback I need to make the course the best it can be)

   (a) Approximately how many hours per partner did you spend on this lab?

   (b) How difficult did you find this lab? (enter a number 1-5, with 5 being very difficult and 1 being very easy)

   (c) Describe the biggest challenge you faced on this lab.

[1] other choices of $p > 1/2$ and $k = c\log(n)$ will work—you might want to play around with the constants to make the analysis as simple as possible, e.g. by setting $p = 2/3$ and/or $k = \frac{\log(n)}{10}$