CS49/Math59 Lab 3

This lab assignment is due **before the start of class** on Wednesday, October 7. Homework handed in during class but after I begin the lecture will be counted as late submissions. Some things to note:

- I encourage you to write your solution using LaTeX, but you are not required to.
- You may have one partner for this assignment, but are not required to. If you work with a partner, submit just one writeup.
- Aside from your partner, you should not discuss problems in detail with anyone. It's OK to discuss approaches at a high level. In fact, I encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.
- In this problem set, you might need an estimate for $n!$. The following asymptotic estimate is called *Stirling's Formula.*

  $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

  Note: it will likely be easier to use a cruder, less accurate bound than Stirling's approximation. For example, from Stirling's Formula, it is easy to see that $n! > (n/e)^n$ for all large n.

Make sure your names are on your submission, and show your work to maximize partial credit.

1. **Analysis.** In class, we saw that if $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$, then there is a tournament on n players with property S_k. (i.e. for each group of k players, there is another player that beats them all.)

   Let $f(k)$ be the least n that satisfies property S_k. Use the probabilistic method result to show that $f(k) = O(k^2 2^k)$. In other words, show that there is an $n = ck^2 2^k$ for some constant c such that the inequality holds.

2. **Increasing Sequences, Decreasing Sequences.** A *subsequence* of a sequence $a = (a_1, \ldots, a_n)$ is a sequence that can be obtained from a by deleting items without reordering. For example, $a_3 a_4 a_n$ is a subsequence of a, and $bdek$ is a subsequence of $abcdefghijk$.

   For a sequence of numbers, an *increasing* subsequence is a subsequence which is sorted in ascending order. For example, $1, 3, 7, 8$ is an increasing subsequence of $2, 1, 9, 3, 7, 4, 8, 12$. Similarly, a *decreasing* subsequence is a subsequence which is sorted in descending order. For some applications, it is useful to try to *avoid* long increasing or decreasing subsequences.

   Let $[n]$ denote the set of integers $\{1, \ldots, n\}$. Show there is a constant c and a way to permute $[n]$ so that the longest increasing or decreasing subsequence in the permuted sequence is less than k, for $k = c\sqrt{n}$. For example, we can permute $[4]$ to get the sequence $(2, 4, 1, 3)$, which has no increasing or decreasing sequence of length 3.

   Try to find the smallest possible constant c.

3. **3sum-free Integers** A set of integers A is is called *3sum-free* if there are no $a_1, a_2, a_3, a_4 \in A$ such that $a_1 + a_2 + a_3 = a_4$. Show that there is a constant $c > 0$ such that every set B of n nonzero integers contains a *3sum-free* set A of size $|A| > cn$. Try to find the largest possible constant c.

4. **Sparse Graphs with Near-Cliques.**

   On a future lab assignment, you will solve the following problem:

   (Alon and Spencer, problem 1.5) Let G be a graph on $n \geq 10$ vertices, and suppose that if we add to G any edge not in G, the number of cliques on ten vertices increases. Show that the number of edges in G is at least $8n - 36$.

   For this lab, you will try to build some intuition. Say (i, j) is a *non-edge* if $(i, j) \notin G$. Call a set of ten vertices S a *near-clique* if it contains exactly one non-edge. Say that a nonedge (i, j) belongs to a near-clique S if $i, j \in S$. Call a graph G "near-clique-full" if every non-edge belongs to at least one near-clique.

   The Alon/Spencer problem above asks you to show that any near-clique-full graph has at least $8n - 36$ edges.

   (a) Consider a near-clique-full graph on 10 vertices. How many edges does it have?

   (b) Describe a near-clique-full graph on n vertices with exactly $8n - 36$ edges.

   (c) Suppose G contains non-edges (i, j) and (i', j') that belong to near-cliques S and S' respectively. What is the most number of edges that can belong to both S and S'? What is the least number of edges that belong to S and S'?

5. **Attribution.** Did you get assistance on any of the problems on this assignment from anyone aside from me and/or your lab partner? For example, did you discuss any problems at a high level with other students? Did you accidentally stumble on solutions while doing a websearch on related material? If so, descibe the nature of the assistance here. (e.g. "We briefly discussed problem 1 with X,Y, and Z" or "We saw a solution on ⟨this website⟩ before finding our own solution") If you (and your partner) worked alone, please say so here.

6. **Lab Questionnaire.** (None of these questions will have an impact on your grade, this is to help provide the feedback I need to make the course the best it can be)

   (a) Approximately how many hours per partner did you spend on this lab?

   (b) How difficult did you find this lab? (enter a number 1-5, with 5 being very difficult and 1 being very easy)

   (c) Describe the biggest challenge you faced on this lab.