CS49/Math59 Lab 2

This lab assignment is due **before the start of class** on Wednesday, September 30. Homework handed in during class but after I begin the lecture will be counted as late submissions. Some things to note:

- I encourage you to write your solution using LaTeX, but you are not required to.
- You may have one partner for this assignment, but are not required to. If you work with a partner, submit just one writeup.
- Aside from your partner, you should not discuss problems in detail with anyone. It's OK to discuss approaches at a high level. In fact, I encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.

Make sure your names are on your submission, and show your work to maximize partial credit.

1. **Random Variables.** It is common to use random variables without explicitly specifying the underlying sample space Ω or the precise function $X : \Omega \to S$. In this problem, you will consider different possible sample spaces for the same random variables, and explore how that affects the independence of the random variables.

   Let X_1, X_2 be random variables each uniform on $\{1, 2, 3, 4, 5, 6\}$.

   (a) Give a sample space Ω, probability distribution P, and definitions for $X_1, X_2 : \Omega \to \{1, \ldots, 6\}$ such that X_1, X_2 are independent.

   (b) Give a sample space Ω, probability distribution P, and definitions for X_1, X_2 such that X_1, X_2 are *not* independent.

2. **Asymptotic Notation.** Prove the following facts from first principles.

   (a) **Transitivity of** $O(\cdot)$. If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.

   (b) **Product Rule**. If $f_1 = O(g_1)$ and $f_2 = O(g_2)$, then $f_1 f_2 = O(g_1 g_2)$.

3. **Asymptotic Properties.** Prove that $f = O(g)$ in each of the following subproblems. Your proofs should be rigorous and can use any facts or properties proven in class or lab, or previously in this assignment. You can also do your proofs from first principles. I provided the solution to the first subproblem below.

   (a) $f(n) = n \log n,\ g(n) = n^2$.
   Solution. We saw in class that $\log n = O(n)$. Using the product rule with $f_1(n) = \log n, f_2(n) = n, g_1(n) = n$, and $g_2(n) = n$, we get that $f(n) = n \log n = f_1 f_2 = O(g_1 g_2) = O(g)$. Thus, $f = O(g)$.

   (b) $f(n) = (\log n)^6,\ g(n) = n^\epsilon$ for some arbitrary constant $\epsilon > 0$.

   (c) $f(n) = 2^{n^3},\ g(n) = 2^{2^n}$.

4. **Asymptotic Analysis.** Order the following functions in ascending order of growth. For example, if $f = O(g)$, place f before g in your ordering. **Note:** you do not need to provide any justification; however, showing at least some of your reasoning will maximize your chances of getting partial credit.

   - $f_1(n) = 3n(\log n)^3$
   - $f_2(n) = n^{4 \log n}$
   - $f_3(n) = 5^n$
   - $f_4(n) = 12n^{7/4}$
   - $f_5(n) = 2^{2^n}$

5. **Probabilistic Method.** In this problem, you will prove the following fact using the probabilistic method.

   Fact 1 *Let n, m be positive integers. Let $0 < p < 1$ and $q := 1 - p$. Then,*

   $$(1 - p^m)^n + (1 - q^n)^m \geq 1 \ .$$

   (a) First, let M be an $n \times m$ matrix with 0/1 entries. Independently fill in each cell $M[i, j]$ with a 1 with probability p, and with a 0 with probability q. Let A_r be the event that row r contains at least one 0 cell, and let B_c be the event that column c contains at least one 1 cell.[1] *Briefly* argue that the events $\{A_r\}$ are mutually independent, as are $\{B_c\}$.

   (b) What is $\Pr[A_r]$? What is $\Pr[B_c]$?

   (c) Now, let $A^* := \bigcap_r A_r$ and $B^* := \bigcap_c B_c$. What is $\Pr[A^*]$? What is $\Pr[B^*]$?

   (d) Next, argue that A^*, B^* cannot both *not* happen, i.e., that $\Pr[A^* \cup B^*] = 1$.

   (e) Using the union bound, conclude the fact must hold.

6. **Attribution.** Did you get assistance on any of the problems on this assignment from anyone aside from me and/or your lab partner? For example, did you discuss any problems at a high level with other students? Did you accidentally stumble on solutions while doing a websearch on related material? If so, descibe the nature of the assistance here. (e.g. "We briefly discussed problem 1 with X,Y, and Z" or "We saw a solution on ⟨this website⟩ before finding our own solution") If you (and your partner) worked alone, please say so here.

7. **Lab Questionnaire.** (None of these questions will have an impact on your grade, this is to help provide the feedback I need to make the course the best it can be)

   (a) Approximately how many hours per partner did you spend on this lab?

   (b) How difficult did you find this lab? (enter a number 1-5, with 5 being very difficult and 1 being very easy)

   (c) Describe the biggest challenge you faced on this lab.

[1] Formally, A_r is the event that $M[r, c] = 0$ for some c. B_c is the event that $M[r, c] = 1$ for some r.