CS49/Math59 Lab 1

This lab assignment is due **before the start of class** on Wednesday, September 16. Homework handed in during class but after I begin the lecture will be counted as late submissions. Some things to note:

- I encourage you to write your solution using LaTeX, but you are not required to.
- You may have one partner for this assignment, but are not required to. If you work with a partner, submit just one writeup.
- Aside from your partner, you should not discuss problems in detail with anyone. It's OK to discuss approaches at a high level. In fact, I encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else.

Make sure your names are on your submission.

1. **Conditional Probability.** Let $A, B_1, \ldots, B_k$ be events such that $\{B_1, \ldots, B_k\}$ partition[1] Ω. Show that
$$\Pr[A] = \sum_{i=1}^{k} \Pr[B_i] \Pr[A|B_i] \ .$$

2. **The Chain Rule (Shoup, exercise 8.5)** For events $A_1, \ldots, A_n$, define $\alpha_1 := \Pr[A_1]$, and for $i = 2, \ldots, n$, define $\alpha_i := \Pr[A_i | A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$ (assume that $\Pr[A_1 \cap \cdots \cap A_{n-1}] \neq 0$). Show that $\Pr[A_1 \cap \cdots \cap A_n] = \alpha_1 \cdots \alpha_n$.

3. **Probability vs. Conditional Probability.** Let $\Omega := \{1, \ldots, 8\}, A := \{1, 3, 5, 7\}$, and $B := \{1, 2, 3, 4\}$.
   (a) Define a probability distribution P_1 on Ω such that $P_1[A] < P_1[A|B]$.
   (b) Define a probability distribution P_2 on Ω such that $P_2[A] = P_2[A|B]$.
   (c) Define a probability distribution P_3 on Ω such that $P_3[A] > P_3[A|B]$.

4. **Random Variables (Shoup, exercise 8.16).** Let X, Y be independent random variables such that X is uniformly distributed over a set S and Y is uniformly distributed over a set $T \subset S$. Define a third random variable Z as follows: if $X \in T$ then $Z := X$. Otherwise, $Z := Y$. Show that Z is uniformly distributed over T.

5. **Algorithm Analysis.** Order the following functions in ascending order of growth. For example, if $f = O(g)$ then place f before g in your ordering.
   (a) $f_1(n) = 10n$.
   (b) $f_2(n) = 5\sqrt{n} \log n$.
   (c) $f_3(n) = 2^{n/3}$.

[1] a *partition* of Ω is a collection of pairwise disjoint subsets whose union equals Ω. For example, $\{1, 3\}, \{2, 5\}, \{4\}$ partition the set $\{1, 2, 3, 4, 5\}$

(d) $f_4(n) = 2(\log n)^2$.

(e) $f_5(n) = e^n/6$.

6. **Probabilistic Method** Solve one of the following problems:

   - Three hundred students took an exam for CS21. There were eight problems to solve. It is known that each problem was correctly solved by at least 200 students. Show that there must be two students such that every problem was solved by at least one of these two students.
   - There are several circles inside a square of side length 1. The total circumference of all the circles is 10. Show that there is a a line that intersects at least four of the circles.

7. **Attribution.** Did you get assistance on any of the problems on this assignment from anyone aside from me and/or your lab partner? For example, did you discuss any problems at a high level with other students? Did you accidentally stumble on solutions while doing a websearch on related material? If so, descibe the nature of the assistance here. (e.g. "We briefly discussed problem 1 with X,Y, and Z" or "We saw a solution on ⟨this website⟩ before finding our own solution") If you (and your partner) worked alone, please say so here.

8. **Lab Questionnaire.** (None of these questions will have an impact on your grade, this is to help provide the feedback I need to make the course the best it can be)

   (a) Approximately how many hours per partner did you spend on this lab?

   (b) How difficult did you find this lab? (enter a number 1-5, with 5 being very difficult and 1 being very easy)

   (c) Describe the biggest challenge you faced on this lab.