1. **Transitivity of Polynomial-time Reductions.** Show that

\[ \text{if } A \leq_P B \text{ and } B \leq_P C \text{ then } A \leq_P C . \]

**Note:** Provide a complete formal proof.

2. **Optimization vs Decision Problems.** In lab we revisited optimization and decision problems. For most optimization problems, there is an obvious analogue as a decision problem. For example, consider the following problem:

**IS-OPT:** Given a graph \( G = (V, E) \), return the size of the largest independent set in \( G \).

IS-OPT has a natural decision problem, namely **INDEPENDENT-SET.** In fact, every optimization problem can be converted to a decision problem this way.

(a) Show that **INDEPENDENT-SET \leq_P IS-OPT.**

(b) Let \( B \) be an arbitrary optimization problem, and let \( A \) be the decision version of \( B \). Show that

\[ A \leq_P B . \]

(c) Show that **IS-OPT \leq_P INDEPENDENT-SET.**

(d) Let \( B \) be an arbitrary optimization problem, and let \( A \) be the decision version of \( B \). Does it \textit{always} hold that

\[ B \leq_P A ? \]

Answer YES or NO. Justify your response.
3. **Liars and Friars.** To get away from Swarthmore’s wintery springtime, you escape to a tropical island. This island is populated by \( n \) inhabitants; each inhabitant is either a *liar* or a *friar*. A friar always tells the truth, but liars cannot be trusted. You want to find the best place to eat dinner, and you definitely do not want to ask a liar (they will recommend bagel bar at Sharples, or Starbucks), so you would like to identify at least one friar. To help identify a friar, you can pair up any two inhabitants \( A, B \) and ask each to identify the other. If either \( A \) or \( B \) identifies that the other is a liar, then at least one of \( A \) and \( B \) is a liar. If both claim the other is a friar, then either both are liars or both are friars.

(a) Show that if more than \( n/2 \) inhabitants are liars, it is not generally possible to identify a friar. You may assume liars collaborate to convince you they are friars.

(b) Now, suppose that more than \( n/2 \) people are friars. Show that with at most \( \left\lfloor \frac{n}{2} \right\rfloor \) pairwise comparisons, it is possible to reduce the problem to one of nearly half the original size. **Hint:** the algorithmic goal of this part is to produce an algorithm that, given \( n \) inhabitants (over half of whom tell the truth) returns a list of \( \approx n/2 \) inhabitants, over half of whom tell the truth.

(c) Use your solution to Part (3b) to construct an algorithm which identifies a single friar.

(d) Show how to find all friars, assuming that more than half of the \( n \) inhabitants are friars.

**Note:** Do not make any assumptions about \( n \), e.g., do not assume \( n \) is odd, or \( n \) is a power of two, etc.