CS41 Homework 6

This homework is due at 11:59PM on Sunday, October 18. Write your solution using LaTeX. Submit this homework using **github** as a file called **hw4.tex**. This is a **partnered homework**. It's ok to discuss approaches at a high level with others; however, you should **primarily discuss approaches with your homework partner.**

If you do discuss problems with others, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you've done with a lab partner *while in lab.* In this case, note (in your **homework submission poll**) who you've worked with and what parts were solved during lab.

1. **Changing edge costs** (Kleinberg and Tardos, 4.2)
   For each of the following two statements, decide whether it is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

   (a) Suppose we are given an instance of the Minimum Spanning Tree (MST) problem on a graph $G = (V, E)$, with edge costs that are all positive and distinct. Let T be a minimum spanning tree for this instance. Now, suppose we replace each cost c_e by its square c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

   True or False? T must still be a minimum spanning tree for this new instance.

   (b) Suppose we are given an instance of the Shortest $s \rightsquigarrow t$ Path Problem on a directed graph $G = (V, E)$, with edge costs that are all positive and distinct. Let P be a a minimum-cost $s \rightsquigarrow t$ path for this instance. Now, suppose we replace each edge cost c_e by its square c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

   True or False? P must still be a minimum $s \rightsquigarrow t$ path for this new instance.

2. **Cell service on the Appalachian Trail**
   The Appalachian Trail is an approximately 2100 mile trail that runs north-south from Maine to Georgia. Recently, several hikers have started to demand cell service along the trail. Other hikers object, assuming that cell phones will ruin the hiking experience. Leaders of the Appalachian Trail Conservatory, who manage the trail, have decided on a compromise – they plan to install cell phone base stations, but only for service at one of the camground areas on the trail. You've been hired to help decide where to place the base stations. Your goal is to place cell phone base stations at certain points on the trail, so that every campground is within four miles of the base stations.

   Give an efficient algorithm that achieves this goal, using as few base stations as possible. The input to your algorithm should be a List of campground locations. The output of your algorithm should be a List of base station locations, so that for each campground there is a base station within four miles of the campground.

3. **Making change**

   Consider the problem of making change for n cents out of the fewest number of coins. Assume that n and the coin values are positive integers (cents).

(a) Describe a greedy algorithm to solve the problem using the US coin denominations of quarters (25), dimes (10), nickels (5), and pennies (1). Prove your algorithm is optimal.

(b) Suppose the country of Algorithmland uses denominations that are powers of c for some integer c. This country uses $k+1$ denominations of $c^0, c^1, \ldots, c^k$. Show that your greedy algorithm works in Algorithmland as well.

(c) Design a currency system of your choosing with at least three coin denominations such that your greedy algorithm does *not* yield a minimum number of coins for some amount of m cents. Assume that one of your denominations has value one, so a solution exists for all values of m.

4. **Extra challenge.** One reason why the United States has the coin denominations it has is to make it easier to make change using a small number of coins. However, is the US set of coin denominations the optimal set of denominations?

   Perhaps some other set of denominations is better. In this problem, design and analyze an algorithm to determine the optimal set of coin denominations. Given an integer N, say the *change efficiency* of a set of denominations of coins as the average number of coins required to make change, where the average is taken over all $1 \leq m \leq N$.

   For example, using the standard US coin denominations $\{1, 5, 10, 25\}$, the coin efficiency given $N = 5$ is 2.2, since it takes m pennies to make change for values $1 \leq m \leq 4$, and one nickel to make change for value $m = 5$.

   Design and analyze an efficient algorithm that takes as input positive integers N and k and returns a set of k denominations that minimizes the coin efficiency for N.