**Bspline Construction Summary**

![Bspline Construction Graph]

**Objective of Bspline interpolation:**
Given the points \((t_0,p_0), (t_1,p_1), (t_2,p_2), (t_3,p_3), (t_4,p_4), (t_5,p_5), ... (t_m,p_m)\), find the coefficients, \(c_j, j = 0 \text{ to } m+n-1\) of the Bspline basis functions, \(N_j^n(t)\), such that the sum,

\[
f(t) = \sum_{j=0}^{m+n-1} c_j N_j^n(t)
\]

produces an interpolating spline of degree \(n\) with values, \(f(t) = p_j\) when \(t = t_i, i = 0 \text{ to } m\).

\(m\) also represents the number of curve segments between \(t_0\) between \(t_m\), which in the example shown above is 6.

**Determining the coefficients, \(c_j\), of an interpolating Bspline:**

**Step 1.** Assemble the knot vector of the form \(\lambda = [\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{m+2n}]\) containing the values of \(t\) where the polynomial curve segments of degree \(n\) are to be joined.

**Note:** Since \(n+1\) Bspline basis functions \(N_j^n(t)\) of degree \(n\) are required to be computed over each interval \(t \in [t_j, t_{j+1}]\), \(n\) additional knot points must be defined to the left of \(t_0\) and to the right of \(t_m\) in order to enable \(f(t)\) to be computed for the intervals \(t \in [t_0, t_1] \text{ and } t \in [t_{m-1}, t_m]\).
Additional left side knots: \( t_{-k} = t_0 - k(t_1 - t_0) \) \( k = 1 \) to \( n \)

Additional right side knots: \( t_{m+k} = t_m + k(t_{m+1} - t_m) \) \( k = 1 \) to \( n \)

When \( n = 3 \), this results in a knot vector of the form

\[
\lambda = \left[ \begin{array}{cccccccc}
 t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 & \cdots & t_m & t_{m+1} & t_{m+2} & t_{m+3}
\end{array} \right] = \left[ \begin{array}{cccc}
 \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{m+2n}
\end{array} \right]
\]

When \( n = 1 \), the knot vector would be of the form

\[
\lambda = \left[ \begin{array}{cccccccc}
 t_{-1} & t_0 & t_1 & t_2 & \cdots & t_m & t_{m+1}
\end{array} \right] = \left[ \begin{array}{cccc}
 \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{m+2n}
\end{array} \right]
\]

For example, given the seven points \((t_i, p_i), i=0\) to 6 shown on the figure of the first page,

For the cubic case \((n=3)\) the knot vector, \( \lambda \), would take the form,

\[
\lambda = \left[ \begin{array}{cccccccc}
 -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18
\end{array} \right] = \left[ \begin{array}{cccc}
 \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{12}
\end{array} \right]
\]

For the linear case \((n=1)\) the knot vector, \( \lambda \), would take the form,

\[
\lambda = \left[ \begin{array}{cccccccc}
 -2 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14
\end{array} \right] = \left[ \begin{array}{cccc}
 \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{8}
\end{array} \right]
\]

**Step 2.** For each \( i (i = 0 \) to \( m \)), compute the set of \( N_j^n(t_i) \) that are nonzero over the interval \( t_i \leq t \leq t_{i+1} \) using the recurrence equation,

\[
N_j^n(t) = \frac{(t - \lambda_j)}{\lambda_{j+n} - \lambda_j} N_j^{n-1}(t) + \frac{(\lambda_{j+n+1} - t)}{\lambda_{j+n+1} - \lambda_{j+1}} N_{j+1}^{n-1}(t)
\]

where \( N_j^0(t) = \begin{cases} 
1 & \text{for } \lambda_j \leq t < \lambda_{j+1} \\
0 & \text{otherwise}
\end{cases} \)

and \( N_j^n(t_i) = 0 \) when \( t < \lambda_j \) or \( t > \lambda_{j+n+1} \)
For example, in the cubic case (i.e. $n = 3$) when $6 \leq t < 8$ (i.e. $t_3 \leq t < t_4$), it can be determined that $j = 6$, since $j$ is found from the interval in the knot vector where $\lambda_j \leq t < \lambda_{j+1}$, which for this case is $\lambda_6 \leq t < \lambda_7$. From the definition of the recurrence equation above, and knowing that $j=6$ allows the recurrence equation to be initialized with $N_6^0(t) = 1$ and used to compute the B-splines of degree $n=1$ to 3 that are nonzero over the interval $\lambda_6 \leq t < \lambda_7$.

As above, for the case where $t_3 \leq t < t_4$ (i.e. $\lambda_6 \leq t < \lambda_7$) it follows that:

\[ n = 0: \quad N_6^0(t) = 1 \]
\[ N_j^0(t) = 0 \quad \text{for all other } j \quad (\text{i.e. except } j = 6) \]

\[ n = 1: \]
\[ N_6^1(t) = \frac{(t - \lambda_0)}{(\lambda_6 - \lambda_0)} N_6^0(t) + \frac{(\lambda_8 - t)}{(\lambda_8 - \lambda_7)} N_7^0(t) = \frac{(t - \lambda_0)}{(\lambda_6 - \lambda_0)} N_6^0(t) \]
\[ N_5^1(t) = \frac{(t - \lambda_3)}{(\lambda_6 - \lambda_3)} N_5^0(t) + \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_6)} N_6^0(t_j) = \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_6)} N_6^0(t) \]
\[ N_j^1(t) = 0 \quad \text{for all other } j \quad (\text{i.e. except } j = 6 \text{ and } 5) \]

\[ n = 2: \]
\[ N_6^2(t) = \frac{(t - \lambda_0)}{(\lambda_6 - \lambda_0)} N_6^1(t) + \frac{(\lambda_9 - t)}{(\lambda_9 - \lambda_7)} N_7^1(t) = \frac{(t - \lambda_0)}{(\lambda_6 - \lambda_0)} N_6^1(t) \]
\[ N_5^2(t) = \frac{(t - \lambda_3)}{(\lambda_5 - \lambda_3)} N_5^1(t) + \frac{(\lambda_8 - t)}{(\lambda_8 - \lambda_6)} N_6^1(t) \]
\[ N_4^2(t) = \frac{(t - \lambda_4)}{(\lambda_4 - \lambda_4)} N_4^1(t) + \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_8)} N_5^1(t) = \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_8)} N_5^1(t) \]
\[ N_j^2(t) = 0 \quad \text{for all other } j \quad (\text{i.e. except } j = 6, 5 \text{ and } 4) \]

\[ n = 3: \]
\[ N_6^3(t) = \frac{(t - \lambda_0)}{(\lambda_6 - \lambda_0)} N_6^2(t) + \frac{(\lambda_9 - t)}{(\lambda_9 - \lambda_7)} N_7^2(t) = \frac{(t - \lambda_0)}{(\lambda_6 - \lambda_0)} N_6^2(t) \]
\[ N_5^3(t) = \frac{(t - \lambda_3)}{(\lambda_5 - \lambda_3)} N_5^2(t) + \frac{(\lambda_8 - t)}{(\lambda_8 - \lambda_6)} N_6^2(t) \]
\[ N_4^3(t) = \frac{(t - \lambda_4)}{(\lambda_4 - \lambda_4)} N_4^2(t) + \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_5)} N_5^2(t) \]
\[ N_3^3(t) = \frac{(t - \lambda_3)}{(\lambda_3 - \lambda_3)} N_3^2(t) + \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_4)} N_4^2(t) = \frac{(\lambda_7 - t)}{(\lambda_7 - \lambda_4)} N_4^2(t) \]
\[ N_j^3(t) = 0 \quad \text{for all other } j \quad (\text{i.e. except } j = 6, 5, 4 \text{ and } 3) \]
Step 3. Compute the 2nd derivatives of the Bspline basis functions, \( N_j^n(t) \), at \( t_0 \) and \( t_m \) using the recurrence equation,

\[
^{(i)}N_j^n(t) = n \left( \frac{1}{\lambda_{j+n} - \lambda_j} \right)^{(i-1)} N_{j}^{n-1}(t) - \frac{1}{\lambda_{j+n+1} - \lambda_j} \right)^{(i-1)} N_{j+1}^{n-1}(t) \right) \]

where \( N_j^0(t) = \begin{cases} 1 & \text{for } \lambda_j \leq t < \lambda_{j+1} \\ 0 & \text{otherwise} \end{cases} \) and \( \ell = \ell^{th} \) derivative

First compute \( N_j^n(t_0) \), \( n = 0 \) to 3 where \( j = 0 \) to 3 using the standard Bspline recurrence equation. It turns out that when \( t = t_0 \), \( N_j^n(t_0) = 0 \) for \( n = 1 \) to 3. With knowledge of the values of \( N_j^n(t_0) \) in hand, the first derivative of \( N_j^n(t_0) \) can be computed as follows:

First derivative of \( n=2 \) Bspline basis functions at \( t_0 \) (i.e. case \( n = 2, \ell = 1 \)):

\[
^{(1)}N_3^2(t_0) = 2 \left( \frac{1}{\lambda_5 - \lambda_3} \right) N_3^1(t_0) - \frac{1}{\lambda_6 - \lambda_4} \right) N_4^1(t_0) = \frac{2}{\lambda_5 - \lambda_3} N_3^1(t_0) = 0
\]

\[
^{(1)}N_2^2(t_0) = 2 \left( \frac{1}{\lambda_4 - \lambda_2} \right) N_2^1(t_0) - \frac{1}{\lambda_5 - \lambda_3} \right) N_3^1(t_0) = \frac{2}{\lambda_4 - \lambda_2} N_2^1(t_0)
\]

\[
^{(1)}N_1^2(t_0) = 2 \left( \frac{1}{\lambda_3 - \lambda_1} \right) N_1^1(t_0) - \frac{1}{\lambda_4 - \lambda_2} \right) N_2^1(t_0) = -\frac{2}{\lambda_4 - \lambda_2} N_2^1(t_0)
\]

Knowing \( ^{(1)}N_3^2(t_0) \), \( ^{(1)}N_2^2(t_0) \), \( ^{(1)}N_1^2(t_0) \) enables the second derivative of the \( n=3 \) (i.e. cubic) Bsplines to be computed as follows:

\[
n = 3, \ell = 2: \quad ^{(2)}N_3^3(t_0) = 3 \left( \frac{1}{\lambda_6 - \lambda_4} \right) ^{(1)}N_3^2(t_0) - \frac{1}{\lambda_5 - \lambda_3} \right) ^{(1)}N_4^2(t_0) = 0
\]

\[
^{(2)}N_2^3(t_0) = 3 \left( \frac{1}{\lambda_5 - \lambda_3} \right) ^{(1)}N_2^2(t_0) - \frac{1}{\lambda_6 - \lambda_4} \right) ^{(1)}N_3^2(t_0)
\]

\[
^{(2)}N_1^3(t_0) = 3 \left( \frac{1}{\lambda_4 - \lambda_2} \right) ^{(1)}N_1^2(t_0) - \frac{1}{\lambda_5 - \lambda_3} \right) ^{(1)}N_2^2(t_0)
\]

\[
^{(2)}N_0^3(t_0) = 3 \left( \frac{1}{\lambda_3 - \lambda_1} \right) ^{(1)}N_0^2(t_0) - \frac{1}{\lambda_4 - \lambda_2} \right) ^{(1)}N_1^2(t_0)
\]

The values of \( ^{(2)}N_j^n(t_m) \) can be computed in a similar fashion.
Step 4. Set up the vector matrix equation of the form,

\[
Ac = d
\]

and solve for the coefficient vector \( c = A^{-1}d \) where,

- \( A \) is a matrix of dimension \((m+n) \times (m+n)\) containing the B spline basis function values (and derivatives) evaluated at each \( t_i, i = 0 \) to \( m \),
- \( c \) is a coefficient vector of dimension \((m+n) \times 1\), and
- \( d \) is a vector of dimension \((m+n) \times 1\) containing the desired values of the spline at the points \( p_i, i = 0 \) to \( m \) (and possibly 1st or 2nd derivative values at points \( t_0 \) and \( t_m \)).
For the cubic case ($n = 3$) with natural spline endpoint conditions (i.e. second derivative = 0 at $t_0$ and $t_m$), the matrix $A$ takes the banded form (where all entries other than the ones shown are assumed to be zero):

$$
\begin{bmatrix}
N_0^3(t_0) & (2)N_1^3(t_0) & (2)N_2^3(t_0) & (2)N_3^3(t_0) \\
N_0^3(t_0) & N_1^3(t_0) & N_2^3(t_0) & N_3^3(t_0) \\
N_1^3(t_1) & N_2^3(t_1) & N_3^3(t_1) & N_4^3(t_1) \\
N_2^3(t_2) & N_3^3(t_2) & N_4^3(t_2) & N_5^3(t_2) \\
& & &
\end{bmatrix}
$$

$$
A = \begin{bmatrix}
(2)N_0^3(t_m) & (2)N_1^3(t_m) & (2)N_2^3(t_m) & (2)N_3^3(t_m) \\
N_{m-3}^3(t_m) & N_{m-2}^3(t_m) & N_{m-1}^3(t_m) & N_m^3(t_m) \\
N_{m-2}^3(t_m) & N_{m-1}^3(t_m) & N_m^3(t_m) & N_{m+1}^3(t_m) \\
N_{m-1}^3(t_m) & N_m^3(t_m) & N_{m+1}^3(t_m) & N_{m+2}^3(t_m) \\
(2)N_{m-1}^3(t_m) & (2)N_m^3(t_m) & (2)N_{m+1}^3(t_m) & (2)N_{m+2}^3(t_m)
\end{bmatrix}
$$

$\Rightarrow$ when $n = 3$ the matrix $A$ has $(m+3)$ rows and $(m+3)$ columns.

The vector $d$ takes the form,

$$d = \begin{bmatrix} 0 & p_0 & p_1 & \cdots & p_m & 0 \end{bmatrix}^T$$

and the coefficient vector $c$ is determined from the equation,

$$c = A^{-1}d \quad \text{where } A^{-1} \text{ is the inverse of the matrix } A.$$
For example, given the seven points \((t_i, p_i), i=0 \text{ to } 6\) shown on the figure of the first page with values,

\[
p_0=(0, 1), \ p_2=(2, 2), \ p_3=(4, 2.5), \ p_4=(6, 1.5), \ p_5=(8, 0.5), \ p_6=(10, 0.25)
\]

and the associated knot array,

\[
\lambda = [-6 \ -4 \ -2 \ 0 \ 2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18]
\]

\[
= [\lambda_0 \ \lambda_1 \ \lambda_2 \ \ldots \ \lambda_{12}]
\]

it follows that,

\[
d = [0 \ 1 \ 2 \ 2.5 \ 1.5 \ 0.5 \ 0.25 \ 0 \ 0]^{T}
\]

\[
A = \begin{bmatrix}
0.25 & -0.5 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.167 & 0.667 & 0.167 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.167 & 0.667 & 0.167 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.167 & 0.667 & 0.167 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.167 & 0.667 & 0.167 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.167 & 0.667 & 0.167 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.167 & 0.667 & 0.167 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.25 & -0.5 & 0.25
\end{bmatrix}
\]

and

\[
c = A^{-1}d = [-0.03 \ 1.0 \ 2.0301 \ 2.8795 \ 1.4519 \ 0.3128 \ 0.2968 \ 0 \ -0.297]^{T}
\]
Computing the Interpolating Bspline function f(t):

Since there are only \( n+1 \) nonzero basis splines for any value of \( t \), the interpolating Bspline function \( f(t) = \sum_{j=0}^{m+n-1} c_j N_j^n(t) \) can be simplified to,

\[
f(t) = \sum_{k=0}^{n} c_{j+k} N_{j+k}^n(t)
\]

where \( j \) is associated with the interval in which \( \lambda_j \leq t < \lambda_{j+1} \).

**Step 1.** Given the knot vector \( \lambda \) and a value for \( t \) find the value of \( j \) such that \( \lambda_j \leq t < \lambda_{j+1} \).

**Step 2.** Given the value of \( j \) determined in Step 1 and the value of \( t \), compute the \((n+1)\) Bspline basis functions \( N_j^n(t) \) that are nonzero over the interval \( \lambda_j \leq t < \lambda_{j+1} \) using the Bspline recurrence equation described previously. For example, when \( t=7 \) the nonzero Bspline basis functions \( N_j^n(t) \) up to degree \( n=3 \) that are nonzero over the interval \( \lambda_6 \leq t < \lambda_7 \) (i.e. \( 6 \leq t < 8 \)) would be found to be,

\[
N_0^0 = 1 \\
N_1^0 = \frac{1}{2} \\
N_2^0 = \frac{1}{4} \\
N_3^0 = \frac{1}{8} \\
N_4^0 = \frac{1}{16} \\
N_5^0 = \frac{1}{32} \\
N_6^0 = \frac{1}{64}
\]

**Step 3.** Assuming the coefficient vector \( c \) has been previously determined, for the cubic case \( (n=3) \) the B spline function \( f(t) \) can be evaluated at \( t=7 \) as,

\[
f(t) = c_{j-3} N_{j-3}^3(t) + c_{j-2} N_{j-2}^3(t) + c_{j-1} N_{j-1}^3(t) + c_j N_j^3(t)
\]

where \( j \) is the index value determined in Step 2.

As shown in the example of Step 2 when \( t=7, \ j = 6 \) so it follows that,

\[
f(t) = c_3 N_3^3(t) + c_4 N_4^3(t) + c_5 N_5^3(t) + c_6 N_6^3(t)
\]

where values for the basis splines \( N_3^3(t=7) = \frac{1}{48}, N_4^3(t=7) = \frac{23}{48}, N_5^3(t=7) = \frac{23}{48}, N_6^3(t=7) = \frac{1}{48} \) at \( t=7 \) are obtained as shown in Step 2.