CS46 practice problems 2

These practice problems are an opportunity for discussion and trying many different solutions. They are **not counted towards your grade**, and **you do not have to submit your solutions.** You are welcome to consider these problems in any order. The later problems require more discussion. The purpose of these problems is to get more comfortable with set notation, thinking about sets, making proof-like arguments, and pondering the mathematical mysteries of infinity.

1. Consider two sets A and B. Using direct proof, show that

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

   (This proof is in the textbook reading, so if you are stuck, refer to that.)

2. Find the problems in the following proofs.

   (a)

   Claim 1. *It turns out that* $1 = 0$.

   Proof. (directly)
   Let x and y be any two non-zero numbers such that $x = y$. Then:

$$
\begin{aligned}
x &= y && \text{our starting assumption} \\
x^2 &= x \cdot y && \text{multiplying by } x \text{ on both sides} \\
x^2 - y^2 &= xy - y^2 && \text{subtracting } y^2 \text{ from both sides} \\
(x+y)\cdot(x-y) &= (x-y)\cdot y && \text{factoring} \\
(x+y) &= y && \text{dividing by } x-y \text{ on both sides} \\
y + y &= y && \text{substituting since } x = y \\
2y &= y && \\
2 &= 1 && \text{since } y \text{ was nonzero} \\
1 &= 0 && \text{subtracting 1 from both sides}
\end{aligned}
$$

   □

   (b)

   Claim 2. *Linear search is* $O(1)$ *runtime.*

   Proof. (by induction on n, the length of the list we are searching)
   base case: If $n = 1$, then linear searching a list of length $n = 1$ takes 1 operation, which is $O(1)$.
   inductive hypothesis: Assume that if we have a list of length k some constant, then linear search on that list takes $O(1)$ time.
   inductive step: Consider a list of length $n = k + 1$. When we do linear search on it, we first do linear search on the list of length k that is the first "part" (the first k elements)

of this list. This takes $O(1)$, according to the inductive hypothesis. If we haven't found the item we're looking for yet, then we do one more check to see if it is the last item in the list. This check takes $O(1)$.

Thus total runtime is $O(1) + O(1) = O(1)$. □

Hint: Recall that an algorithm has runtime $O(f(n))$ if there are values of $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ with $c > 0$ and $n_0 > 0$ such that for all $n \geq n_0$, the runtime for input of size n is at most $c \cdot f(n)$.

(c)

Claim 3. *In any set of h horses, all horses are the same color.*

Proof. (by induction)

Base case: For $h = 1$, in any set containing just one horse, clearly all horses have the same color.

Induction hypothesis: Assume that in any set containing k horses, all horses have the same color.

Induction step: Consider any set H of $h = k + 1$ horses. We want to show that every horse in this set is the same color.

Remove one horse from this set to get a set H_1 of k horses. By the inductive hypothesis, every horse in this set is the same color.

Now replace the removed horse, and remove a *different* horse. Now we have a set H_2 of k horses. By the inductive hypothesis, every horse in *this* set is the same color.

Therefore, all horses in the set H are the same color. □

3. We saw in class that $|\mathbb{N}| = |\mathbb{Z}| = \aleph_0 \neq |\mathbb{R}| = \aleph_1$, even though both $\mathbb{N}$ and $\mathbb{R}$ are infinite sets. In this problem, we will consider the set $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}\}$ of all rational numbers. We will argue that $\mathbb{Q}$ is "countable", too.

Definition 4. *A set S is called* **countable** *if S is finite or has the same cardinality as $\mathbb{N}$.*

The idea is that we can "count" the elements of S according to their pairing with the elements of $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. We will also use the term "enumerable" for this property (for the same reason: we can enumerate the elements of S).

(a) Draw a Venn diagram of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$. Where does $\mathbb{Q}$ fit in this diagram?

(b) Let's start by only considering the positive rational numbers $\mathbb{Q}^+ = \mathbb{Q} \cap \{x \mid x \geq 0\}$. Come up with a way to list the elements of $\mathbb{Q}^+$. Your list is allowed to have duplicates.

Hint: The definition of $\mathbb{Q}$ says that every rational number $x \in \mathbb{Q}$ is representable as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$. How can you use this to make sure your list includes every $x \in \mathbb{Q}$?

(c) Remove duplicates from your list. (You don't need a rigorous description of how to do this, but you should consider how you would identify duplicated numbers and make sure that you don't eliminate some number completely from the list.)

(d) Congratulations! Now that you have a list, you can set up your function $f : \mathbb{N} \to \mathbb{Q}^+$ and check that it is one-to-one and onto.

(e) To finish the argument, explain how to extend our function f which shows that $\mathbb{Q}^+$ is countable to a function f' which shows that all of $\mathbb{Q}$ (including the negative rational numbers) is countable.

Hint: We saw a "trick" for dealing with positive/negative numbers in the proof that $\mathbb{Z}$ is countable in class. Try a similar technique here.

4. Prove that $\mathbb{N} \times \mathbb{N}$ is countable. (**Hint:** follow a similar structure as what you used for showing $\mathbb{Q}$ is countable.)