CS46, Swarthmore College, Spring 2014
Homework 6 – due 8 April
Your Name(s) Here

1. Sipser 4.31: Say that a variable A in a CFL G is **usable** if it appears in some derivation of some string $w \in G$. Given a context free grammar G and variable A, consider the problem of testing if A is usable.

   (a) Formulate this problem as a language $USABLE_A$

   (b) Show that $USABLE_A$ is decidable

2. Sipser 5.13: A **useless state** in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that it is undecidable. Hint: Is it possible to decide if a machine M halts on any string w?

3. Sipser 5.20: Prove that there exists an undecidable subset of $ONES = \{1\}^*$, the set of all strings on a unary alphabet. One approach is to construct a language $L \subset ONES$ and show it is undecidable. Another approach is to prove that L must exist without needing to explicitly construct L.

4. Sipser 5.16: Let $\Gamma = \{a, b, \sqcup\}$ be the tape alphabet for all TMs in this problem. For each value of an integer $k \geq 2$, consider all k-state TMs that halt when started with a blank tape. Let $CC(k)$ be the maximum number of as that remain on the tape of all TMs with k-states. Note that since there a finite number of k-state Turing machines for each value of k, $CC(k)$ is well defined for each k. We call $CC : \mathcal{N} \rightarrow \mathcal{N}$ the crazy-corgi function.

   (a) Show that if $f : \mathcal{N} \rightarrow \mathcal{N}$ is a computable function, then there is some integer q such that $CC(n+q) \geq f(n)$. Hint: design a machine with roughly q states that when started with input $w = a^n$ halts with $a^{f(n)}$ on its tape.

   (b) Show that $CC(n)$ is not computable. Hint: assume by contradiction that h computes $CC(n)$ given input a^n. Show that this implies $h_2(n) = CC(2n)$ is computable. Go from there.