1. Sipser 4.31: Say that a variable $A$ in a CFL $G$ is **usable** if it appears in some derivation of some string $w \in G$. Given a context free grammar $G$ and variable $A$, consider the problem of testing if $A$ is usable.

(a) Formulate this problem as a language $USABLE_A$

(b) Show that $USABLE_A$ is decidable

2. Sipser 5.13: A **useless state** in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that it is undecidable. Hint: Is it possible to decide if a machine $M$ halts on any string $w$?

3. Sipser 5.20: Prove that there exists an undecidable subset of $ONES = \{1\}^*$, the set of all strings on a unary alphabet. One approach is to construct a language $L \subset ONES$ and show it is undecidable. Another approach is to prove that $L$ must exist without needing to explicitly construct $L$.

4. Sipser 5.16: Let $\Gamma = \{a, b, \sqcup\}$ be the tape alphabet for all TMs in this problem. For each value of an integer $k \geq 2$, consider all $k$-state TMs that halt when started with a blank tape. Let $CC(k)$ be the maximum number of $a$s that remain on the tape of all TMs with $k$-states. Note that since there a finite number of $k$-state Turing machines for each value of $k$, $CC(k)$ is well defined for each $k$. We call $CC : \mathcal{N} \to \mathcal{N}$ the crazy-corgi function.

(a) Show that if $f : \mathcal{N} \to \mathcal{N}$ is a computable function, then there is some integer $q$ such that $CC(n + q) \geq f(n)$. Hint: design a machine with roughly $q$ states that when started with input $w = a^n$ halts with $af(n)$ on its tape.

(b) Show that $CC(n)$ is not computable. Hint: assume by contradiction that $h$ computes $CC(n)$ given input $a^n$. Show that this implies $h_2(n) = CC(2n)$ is computable. Go from there.