Part 1: Written homework

1. Let $A$ be a set of finite cardinality. Use induction on the cardinality of $A$ to prove that $|2^A| = 2^{|A|}$, i.e., prove that the cardinality of the power set of $A$ is two to the cardinality of $A$.

2. A natural isomorphism between sets $A$ and $B$ is a simple bijective function, $f : A \rightarrow B$, that can be considered a slight re-writing of elements from either set. For example, let $A = \{a, b, c\}$ and let $B = \{(a), (b), (c)\}$ be a set of singleton tuples. While there are many bijections between $A$ and $B$, the one that maps $x \in A$ to $(x) \in B$ seems “natural”. In addition to the term natural isomorphism, we define for two sets $A$ and $B$ the expression $B^A$ to be the set of all functions from $A$ to $B$. Let $B = \{0, 1\}$, and let $A = \{a, b, c\}$.

(a) Give an example of one element of $B^A$

(b) Write the power set of $A$, $2^A$.

(c) Describe a natural isomorphism between $B^A$ and $2^A$.

3. Consider an infinite grid of white squares. We could assign each grid square $q$ a label $(i, j), i, j \in \mathbb{Z}$ to indicate the square $q$ is located at column $i$, row $j$. On this grid we initially place $n$ garnet squares. We then update every square’s color according to the following rule: for a square $q$ at position $(i, j)$ if at least two of the three squares at positions $(i - 1, j), (i, j), (i, j - 1)$, i.e, the squares at $q$, below $q$ and to the left of $q$ are garnet, we color $q$ garnet. Otherwise we color it white. Prove that if we perform $n$ rounds of this update step, all squares will be white after the final step, regardless of the initial placement of the $n$ squares. Hint: consider the bounding box containing all $n$ garnet squares. What happens to the bounding box over time?

Part 2: Programming Exercise

Write a program that lists all the elements of the power set of $n$ elements in short lexicographical order. You may assume that the elements are labeled $a, b, c, \ldots$ and that $n \leq 26$. Your program should take a single command line argument as the parameter $n$. 